Acta Numerica (1996), *pp.* 159–333

Exact and approximate controllability for distributed parameter systems

R. Glowinski
University of Houston, Houston, Texas, USA
Université Pierre et Marie Curie, Paris, France
C.E.R.F.A.C.S., Toulouse, France

J.L. Lions
Collège de France, Rue d'Ulm, 75005 Paris, France

This is the second part of an article which was started in the previous volume of *Acta Numerica.* References in the text to Section 1 refer to the preceding article.

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2. BOUNDARY CONTROL

2.1. Dirichlet control (I): Formulation of the control problem

We consider again the *state equation*

$$\frac{\partial y}{\partial t} + Ay = 0 \text{ in } Q, \tag{2.1}$$

where the *second-order elliptic operator* A is as in Section 1.1, and where the control v is now a *boundary control of Dirichlet* type, namely

$$y = \begin{cases} v \text{ on } \Sigma_0 = \Gamma_0 \times (0,T), \\ 0 \text{ on } \Sigma \backslash \Sigma_0, \end{cases} \tag{2.2}$$

where Σ_0 is a (regular) subset of Γ.

The *initial condition* is (for simplicity)

$$y(0) = 0. \tag{2.3}$$

In (2.2) we assume that

$$v \in L^2(\Sigma_0). \tag{2.4}$$

Then, assuming that the coefficients of operator A are smooth enough (cf. Lions and Magenes (1968) for precise statements), the parabolic problem (2.1)–(2.3) has a *unique solution* such that

$$y \in L^2(0,T;L^2(\Omega)) \ (= L^2(Q)), \quad \frac{\partial y}{\partial t} \in L^2(0,T;H^{-2}(\Omega)), \tag{2.5}$$

so that

$$y \in C^0([0,T];H^{-1}(\Omega)). \tag{2.6}$$

Remark 2.1 The solution y to (2.1)–(2.3) is defined, as usual, by *transposition.* Properties (2.5) and (2.6) still hold true if $v \in L^2(0,T;H^{-1/2}(\Gamma_0))$ (the notation is that used in Lions and Magenes (1968)).

Concerning *controllability*, the key result is given by the following:

Proposition 2.1 *When v spans $L^2(\Sigma_0)$, the function $y(T;v)$ spans a dense subspace of $H^{-1}(\Omega)$.*

Proof. We shall give a (nonconstructive) proof based on the *Hahn–Banach theorem.* Consider, thus, $f \in H_0^1(\Omega)$ such that

$$\langle y(T;v), f\rangle = 0, \ \forall v \in L^2(\Sigma_0), \tag{2.7}$$

where, in (2.7), $\langle \cdot,\cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$; next, define ψ by

$$-\frac{\partial \psi}{\partial t} + A^*\psi = 0 \text{ in } Q, \quad \psi(T) = f, \ \psi = 0 \text{ on } \Sigma. \tag{2.8}$$

Multiplying both sides of the first equation in (2.8) by the solution $\{x, t\} \to y(x, t; v)$ of problem (2.1)–(2.3) we obtain after integration by parts

$$\langle y(T; v), f \rangle = -\int_{\Sigma_0} \frac{\partial \psi}{\partial n_{A^*}} v \, \mathrm{d}\Gamma \, \mathrm{d}t, \tag{2.9}$$

where $\partial/\partial n_{A^*}$ denotes the *conormal derivative* operator associated with A^* (if $A = A^* = -\Delta$, then $\partial/\partial n_A = \partial/\partial n_{A^*} = \partial/\partial n$ where $\partial/\partial n$ is the usual outward normal derivative operator at Γ). Then (2.7) is equivalent to

$$\partial\psi/\partial n_{A^*} = 0 \text{ on } \Sigma_0. \tag{2.10}$$

It follows from (2.8), (2.10) that the *Cauchy data* of ψ are zero on Σ_0; using again the *Mizohata's uniqueness theorem*, we obtain that $\psi = 0$ in Q, so that $f = 0$, which completes the proof of the proposition. □

We can now formulate the following *approximate controllability problems* (where $\mathrm{d}\Sigma = \mathrm{d}\Gamma \, \mathrm{d}t$):

Problem 1. It is defined by

$$\inf_v \frac{1}{2} \int_{\Sigma_0} v^2 \, \mathrm{d}\Sigma, \; v \in L^2(\Sigma_0), \; y(T; v) \in y_T + \beta B_{-1}, \tag{2.11}$$

where, in (2.11), y_T is given in $H^{-1}(\Omega)$, $\beta > 0$, B_{-1} denotes the unit ball of $H^{-1}(\Omega)$ and $t \to y(t; v)$ is the solution of (2.1)–(2.3) associated with the control v.

Problem 2. It is the variant of problem (2.11) defined by

$$\inf_{v \in L^2(\Sigma_0)} \left[\frac{1}{2} \int_{\Sigma_0} v^2 \, \mathrm{d}\Sigma + \tfrac{1}{2} k \| y(T; v) - y_T \|_{-1}^2 \right], \tag{2.12}$$

where, in (2.12), $k > 0$, y_T and $y(T; v)$ are as in (2.11), and where

$$\begin{cases} \forall y \in H^{-1}(\Omega), \|y\|_{-1} = \left(\int_\Omega |\nabla \varphi|^2 \, \mathrm{d}x \right)^{1/2} \\ \quad \text{with } \varphi \text{ the unique solution in } H_0^1(\Omega) \text{ of} \\ \int_\Omega \nabla\varphi \cdot \nabla\theta \, \mathrm{d}x = \langle y, \theta \rangle, \; \forall \theta \in H_0^1(\Omega). \end{cases}$$

Both problems (2.11) and (2.12) have a unique solution.

2.2. Dirichlet control (II): Optimality conditions and dual formulations

We discuss first problem (2.12) which is simpler than problem (2.11). Let us denote by $J_k(\cdot)$ the cost function in (2.12); using the relation

$$(J_k'(v), w)_{L^2(\Sigma_0)} = \lim_{\substack{\theta \to 0 \\ \theta \neq 0}} \frac{J_k(v + \theta w) - J_k(v)}{\theta}, \; \forall v, w \in L^2(\Sigma_0), \tag{2.13}$$

we can show that

$$(J_k'(v), w)_{L^2(\Sigma_0)} = \int_{\Sigma_0} \left(v - \frac{\partial p}{\partial n_{A^*}}\right) w \, \mathrm{d}\Sigma, \ \forall v, w \in L^2(\Sigma_0), \tag{2.14}$$

where, in (2.14), the *adjoint state function* p is obtained from v via the solution of (2.1)–(2.3) and of the *adjoint state equation*

$$-\frac{\partial p}{\partial t} + A^* p = 0 \text{ in } Q, \quad p = 0 \text{ on } \Sigma,$$
$$p(T) \in H_0^1(\Omega) \text{ and } -\Delta p(T) = k(y(T) - y_T) \text{ in } \Omega. \tag{2.15}$$

Suppose now that u is *the* solution of the control problem (2.12); since $J_k'(u) = 0$, we have then the following optimality system satisfied by u and the corresponding state and adjoint state functions:

$$u = \frac{\partial p}{\partial n_{A^*}}|_{\Sigma_0},$$

$$\frac{\partial y}{\partial t} + Ay = 0 \text{ in } Q, \quad y(0) = 0, \ y = 0 \text{ on } \Sigma\backslash\Sigma_0 \text{ and } y = \frac{\partial p}{\partial n_{A^*}} \text{ on } \Sigma_0,$$

$$-\frac{\partial p}{\partial t} + A^* p = 0 \text{ in } Q, \quad p = 0 \text{ on } \Sigma, \ p(T) = f,$$

where f is the *unique* solution in $H_0^1(\Omega)$ of the Dirichlet problem

$$-\Delta f = k(y(T) - y_T) \text{ in } \Omega, f = 0 \text{ on } \Gamma. \tag{2.16}$$

In order to identify the dual problem of (2.12), we proceed as in the above sections by introducing (in the spirit of the *Hilbert Uniqueness Method*) the operator $\Lambda \in \mathcal{L}(H_0^1(\Omega); \ H^{-1}(\Omega))$ defined by

$$\Lambda \hat{f} = -\hat{\varphi}(T), \ \forall \hat{f} \in H_0^1(\Omega), \tag{2.17}$$

where the function $\hat{\varphi}$ is obtained from $\hat{f}$ as follows:

Solve first

$$-\frac{\partial \hat{\psi}}{\partial t} + A^* \hat{\psi} = 0 \text{ in } Q, \quad \hat{\psi} = 0 \text{ on } \Sigma, \ \hat{\psi}(T) = \hat{f} \tag{2.18}$$

and then,

$$\frac{\partial \hat{\varphi}}{\partial t} + A\hat{\varphi} = 0 \text{ in } Q, \quad \hat{\varphi}(0) = 0, \ \hat{\varphi} = 0 \text{ on } \Sigma\backslash\Sigma_0, \ \hat{\varphi} = \frac{\partial \hat{\psi}}{\partial n_{A^*}} \text{ on } \Sigma_0. \tag{2.19}$$

We can easily show that (with obvious notation)

$$\langle \Lambda f_1, f_2 \rangle = \int_{\Sigma_0} \frac{\partial \psi_1}{\partial n_{A^*}} \frac{\partial \psi_2}{\partial n_{A^*}} \, \mathrm{d}\Gamma \, \mathrm{d}t, \ \forall f_1, \ f_2 \in H_0^1(\Omega). \tag{2.20}$$

It follows from (2.20) that the operator Λ is *self-adjoint* and *positive semi-*

definite; indeed, it follows from *Mizohata's uniqueness theorem* that the operator Λ is *positive definite.* However, the operator Λ *is not an isomorphism* from $H_0^1(\Omega)$ *onto* $H^{-1}(\Omega)$ (implying that, in general, *we do not have exact boundary controllability here*).

Back to (2.16) we observe that from the definition of Λ we have $y(T) = -\Lambda f$, which implies in turn that f is the unique solution in $H_0^1(\Omega)$ of

$$-k^{-1}\Delta f + \Lambda f = -y_T. \tag{2.21}$$

Problem (2.21) is precisely the dual problem we are looking for. From the properties of operator $-k^{-1}\Delta + \Lambda$, problem (2.21) can be solved by a *conjugate gradient algorithm* operating in the space $H_0^1(\Omega)$; we shall return to this issue in Section 2.3.

Let us consider the control problem (2.11); using the *Fenchel–Rockafellar convex duality theory* as in the above sections, we can show that *the* solution u of problem (2.11) is characterized by the following optimality system

$$u = \frac{\partial p}{\partial n_{A^*}}|_{\Sigma_0}, \tag{2.22}$$

$$\frac{\partial y}{\partial t} + Ay = 0 \text{ in } Q, \quad y(0) = 0,\ y = 0 \text{ on } \Sigma\backslash\Sigma_0 \text{ and } y = \frac{\partial p}{\partial n_{A^*}} \text{ on } \Sigma_0, \tag{2.23}$$

$$-\frac{\partial p}{\partial t} + A^*p = 0 \text{ in } Q, \quad p = 0 \text{ on } \Sigma,\ p(T) = f, \tag{2.24}$$

where f is the *unique* solution of the following *variational inequality* (with $\|\hat{f}\|_{H_0^1(\Omega)} = (\int_\Omega |\nabla \hat{f}|^2\,dx)^{1/2}$, $\forall \hat{f} \in H_0^1(\Omega)$):

$$\begin{cases} f \in H_0^1(\Omega), \\ \langle \Lambda f, \hat{f} - f\rangle + \beta\|\hat{f}\|_{H_0^1(\Omega)} - \beta\|f\|_{H_0^1(\Omega)} + \langle y_T, \hat{f} - f\rangle \geq 0,\ \forall \hat{f} \text{ in } H_0^1(\Omega). \end{cases} \tag{2.25}$$

Problem (2.25) is precisely the dual problem to (2.11). The solution of problem (2.25) will be discussed in Section 2.3.

2.3. Dirichlet control (III): Iterative solution of the control problems

2.3.1. Conjugate gradient solution of problem (2.12)

It follows from Section 2.2 that solving the control problem (2.12) is equivalent to solving the linear equation

$$J_k'(u) = 0, \tag{2.26}$$

where operator J_k' is defined by (2.1)–(2.3), (2.14), (2.15). It is fairly easy to show that the *linear* part of operator J_k', namely

$$v \to J_k'(v) - J_k'(0),$$

is *symmetric* and *strongly elliptic* over $L^2(\Sigma_0)$. From these properties, problem (2.26) can be solved by a *conjugate gradient algorithm* operating in the space $L^2(\Sigma_0)$. It follows from Section 1.8.2 that this algorithm is as follows.

Description of the conjugate gradient algorithm:

$$u^0 \text{ is given in } L^2(\Sigma_0); \tag{2.27}$$

solve

$$\frac{\partial y^0}{\partial t} + Ay^0 = 0 \text{ in } Q, \quad y^0(0) = 0,\ y^0 = u^0 \text{ on } \Sigma_0,\ y^0 = 0 \text{ on } \Sigma\backslash\Sigma_0, \tag{2.28}$$

and then

$$\begin{cases} f^0 \in H_0^1(\Omega), \\ -\Delta f^0 = k(y^0(T) - y_T) \text{ in } \Omega, \end{cases} \tag{2.29}$$

and finally

$$-\frac{\partial p^0}{\partial t} + A^* p^0 = 0 \text{ in } Q, \quad p^0 = 0 \text{ on } \Sigma,\ p^0(T) = f^0. \tag{2.30}$$

Set

$$g^0 = u^0 - \frac{\partial p^0}{\partial n_{A^*}}|_{\Sigma_0}, \tag{2.31}$$

and then

$$w^0 = g^0. \tag{2.32}$$

For $n \geq 0$, assuming that u^n, g^n, w^n are known, compute u^{n+1}, g^{n+1}, w^{n+1} as follows:
Solve

$$\frac{\partial \bar{y}^n}{\partial t} + A\bar{y}^n = 0 \text{ in } Q, \quad \bar{y}^n(0) = 0,\ \bar{y}^n = w^n \text{ on } \Sigma_0,\ \bar{y}^n = 0 \text{ on } \Sigma\backslash\Sigma_0, \tag{2.33}$$

and then

$$\begin{cases} \bar{f}^n \in H_0^1(\Omega), \\ -\Delta \bar{f}^n = k\bar{y}^n(T) \text{ in } \Omega, \end{cases} \tag{2.34}$$

and finally

$$-\frac{\partial \bar{p}^n}{\partial t} + A^* \bar{p}^n = 0 \text{ in } Q, \quad \bar{p}^n = 0 \text{ on } \Sigma,\ \bar{p}^n(T) = \bar{f}^n. \tag{2.35}$$

Compute

$$\bar{g}^n = w^n - \frac{\partial \bar{p}^n}{\partial n_{A^*}}|_{\Sigma_0}, \tag{2.36}$$

and then

$$\rho_n = \int_{\Sigma_0} |g^n|^2 \,\mathrm{d}\Gamma\,\mathrm{d}t \Big/ \int_{\Sigma_0} \bar{g}^n w^n \,\mathrm{d}\Gamma\,\mathrm{d}t, \tag{2.37}$$

$$u^{n+1} = u^n - \rho_n w^n, \tag{2.38}$$

$$g^{n+1} = g^n - \rho_n \bar{g}^n. \tag{2.39}$$

If $\|g^{n+1}\|_{L^2(\Sigma_0)}/\|g^0\|_{L^2(\Sigma_0)} \leq \epsilon$ *take* $u = u^{n+1}$, *else compute*

$$\gamma_n = \|g^{n+1}\|^2_{L^2(\Sigma_0)}/\|g^n\|^2_{L^2(\Sigma_0)} \tag{2.40}$$

and update w^n *by*

$$w^{n+1} = g^{n+1} + \gamma_n w^n. \tag{2.41}$$

Do $n = n+1$ *and go to* (2.33).

Remark 2.2 The number of iterations necessary to obtain the convergence varies here too, as $k^{1/2} \ln \epsilon^{-1/2}$.

2.3.2. Conjugate gradient solution of the dual problem (2.21)

We mentioned in Section 2.2 that the *dual problem* (2.21), namely

$$-k^{-1}\Delta f + \Lambda f = -y_T,$$

can be solved by a *conjugate gradient algorithm* operating in the space $H_0^1(\Omega)$; from the definition of operator Λ (see (2.17)–(2.19)), and from Section 1.8.2, this algorithm takes the following form:

$$f^0 \text{ is given in } H_0^1(\Omega); \tag{2.42}$$

solve

$$-\frac{\partial p^0}{\partial t} + A^* p^0 = 0 \text{ in } Q, \quad p^0 = 0 \text{ on } \Sigma,\ p^0(T) = f^0, \tag{2.43}$$

and

$$\frac{\partial y^0}{\partial t} + Ay^0 = 0 \text{ in } Q, \quad y^0(0) = 0,\ y^0 = 0 \text{ on } \Sigma\backslash\Sigma_0,\ y^0 = \frac{\partial p^0}{\partial n_{A^*}} \text{ on } \Sigma_0. \tag{2.44}$$

Solve now

$$\begin{cases} g^0 \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \nabla g^0 \cdot \nabla z \, dx = k^{-1} \int_\Omega \nabla f^0 \cdot \nabla z \, dx + \langle y_T - y^0(T), z \rangle,\ \forall z \in H_0^1(\Omega), \end{cases} \tag{2.45}$$

and set

$$w^0 = g^0. \tag{2.46}$$

Then, for $n \geq 0$, *assuming that* f^n, g^n, w^n *are known, compute* f^{n+1}, g^{n+1}, w^{n+1} *as follows:*
Solve

$$-\frac{\partial \bar{p}^n}{\partial t} + A^* \bar{p}^n = 0 \text{ in } Q, \quad \bar{p}^n = 0 \text{ on } \Sigma,\ \bar{p}^n(T) = w^n, \tag{2.47}$$

and

$$\frac{\partial \bar{y}^n}{\partial t} + A\bar{y}^n = 0 \text{ in } Q, \quad \bar{y}^n(0) = 0, \ \bar{y}^n = 0 \text{ on } \Sigma\backslash\Sigma_0, \ \bar{y}^n = \frac{\partial \bar{p}^n}{\partial n_{A^*}} \text{ on } \Sigma_0. \tag{2.48}$$

Solve now

$$\begin{cases} \bar{g}^n \in H_0^1(\Omega), \\ \int_\Omega \nabla \bar{g}^n \cdot \nabla z \, \mathrm{d}x = k^{-1} \int_\Omega \nabla w^n \cdot \nabla z \, \mathrm{d}x - \langle \bar{y}^n(T), z \rangle, \ \forall z \in H_0^1(\Omega). \end{cases} \tag{2.49}$$

Compute

$$\rho_n = \int_\Omega |\nabla g^n|^2 \, \mathrm{d}x \Big/ \int_\Omega \nabla \bar{g}^n \cdot \nabla w^n \, \mathrm{d}x \tag{2.50}$$

and then

$$f^{n+1} = f^n - \rho_n w^n, \tag{2.51}$$

$$g^{n+1} = g^n - \rho_n \bar{g}^n. \tag{2.52}$$

If $\|g^{n+1}\|_{H_0^1(\Omega)} / \|g^0\|_{H_0^1(\Omega)} \leq \epsilon$ *take* $f = f^{n+1}$ *and solve* (2.24) *to obtain* $u = \partial p / \partial n_{A^*}|_{\Sigma_0}$; *if the above stopping test is not satisfied, compute*

$$\gamma_n = \int_\Omega |\nabla g^{n+1}|^2 \, \mathrm{d}x \Big/ \int_\Omega |\nabla g^n|^2 \, \mathrm{d}x \tag{2.53}$$

and then

$$w^{n+1} = g^{n+1} + \gamma_n w^n. \tag{2.54}$$

Do $n = n + 1$ *and go to* (2.47).

Remark 2.3 Remark 2.2 still holds for algorithm (2.42)–(2.54).

The *finite element* implementation of the above algorithm will be discussed in Section 2.5, while the results of numerical experiments will be presented in Section 2.6.

2.3.3. Iterative solution of problem (2.25)

Problem (2.25) can also be written as

$$-y_T \in \Lambda f + \beta \partial j(f), \tag{2.55}$$

which is a *multivalued* equation in $H^{-1}(\Omega)$, the unknown function f belonging to $H_0^1(\Omega)$; in (2.55), $\partial j(f)$ denotes the *subgradient* at f of the convex function $j : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$j(\hat{f}) = \left(\int_\Omega |\nabla \hat{f}|^2 \, \mathrm{d}x \right)^{1/2}, \ \forall \hat{f} \in H_0^1(\Omega).$$

Problem (2.25), (2.55) is clearly a variant of problem (1.237) (see Section 1.8.8) and as such can be solved by those *operator splitting methods* advocated in Section 1.8.8. To derive these methods we associate with the '*elliptic problem*' (2.55) the following *initial value problem*

$$\begin{cases} \frac{\partial}{\partial \tau}(-\Delta f) + \Lambda f + \beta \partial j(f) = -y_T, \\ f(0) = f_0(\in H_0^1(\Omega)), \end{cases} \tag{2.56}$$

where, in (2.56), τ is a *pseudo-time*.

To capture the steady-state solution of (2.56) (i.e. the solution of problem (2.25), (2.55)) we can approximately integrate (2.56) from $\tau = 0$ to $\tau = +\infty$ by a *Peaceman–Rachford scheme*, like the one described just below:

$$f^0 = f_0 \text{ given in } H_0^1(\Omega); \tag{2.57}$$

then, for $m \geq 0$, *compute* $f^{m+1/2}$ *and* f^{m+1}, *from* f^m, *by solving in* $H_0^1(\Omega)$ *the following problems*:

$$\frac{(-\Delta f^{m+1/2}) - (-\Delta f^m)}{\Delta \tau / 2} + \beta \partial j(f^{m+1/2}) + \Lambda f^m = -y_T, \tag{2.58}$$

and

$$\frac{(-\Delta f^{m+1}) - (-\Delta f^{m+1/2})}{\Delta \tau / 2} + \beta \partial j(f^{m+1/2}) + \Lambda f^{m+1} = -y_T, \tag{2.59}$$

where $\Delta\tau(> 0)$ is a (pseudo) time discretization step.

As in Section 1.8.8, for problem (1.237), the *convergence* of $\{f^m\}_{m \geq 0}$ to the solution of (2.25), (2.55) is a direct consequence of P.L. Lions and B. Mercier (1979), Gabay (1982; 1983) and Glowinski and Le Tallec (1989); the convergence results proved in the above references apply to the present problem since operator Λ (respectively functional $j(\cdot)$) is *linear, continuous and positive definite* (respectively *convex* and *continuous*) over $H_0^1(\Omega)$. As in Section 1.8.8, we can also use a θ-scheme to solve problem (2.25), (2.55); we shall not describe this scheme here since it is a straightforward variant of algorithm (1.242)–(1.245) (actually such an algorithm is described in Carthel *et al.* (1994), where it has been applied to the solution of the boundary control problem (2.11), (2.25) in the particular case where $\Gamma_0 = \Gamma$).

Back to algorithm (2.57)–(2.59) we observe that problem (2.59) can also be written as

$$\frac{-\Delta f^{m+1} + 2\Delta f^{m+1/2} - \Delta f^m}{\Delta \tau / 2} + \Lambda f^{m+1} = \Lambda f^m. \tag{2.60}$$

Problem (2.60) is a particular case of problem (2.21); it can be solved therefore by the conjugate gradient algorithm described in Section 2.3.2. Concerning the solution of problem (2.58), we observe that the solution of a closely related problem (namely problem (1.343) in Section 1.10.4) has already been discussed; since the solution methods for problem (1.343) and

(2.58) are essentially the same we shall not discuss the solution of (2.58) further.

2.4. Dirichlet control (IV): Approximation of the control problems

2.4.1. Generalities and synopsis

It follows from Section 2.3.3 that the solution of the *state constrained* control problem (2.11) (in fact of its dual problem (2.25)) can be reduced to a sequence of problems similar to (2.21), which is itself the dual problem of the control problem (2.12) (where the closeness of $y(T)$ to the target y_T is forced via *penalty*); we shall therefore concentrate our discussion on the approximation of the control problem (2.12), only.

We shall address both the 'direct' solution of problem (2.12) and the solution of the dual problem (2.21).

The notation will be essentially as in Sections 1.8 and 1.10.6.

2.4.2. Time discretization of problems (2.12) and (2.21)

The *time discretization* of problems (2.12) and (2.21) can be achieved using either first-order or second-order accurate time discretization schemes, very close to those already discussed in Sections 1.8 and 1.10.6 (see also Carthel *et al.* (1994, Sections 5 and 6)). Instead of essentially repeating the discussion which took place in the above sections and reference, we shall describe another *second-order accurate* time discretization scheme, recently introduced by Carthel (1994); actually, the numerical results shown in Section 2.6 have been obtained using this new scheme.

The time discretization of the control problem (2.12) is defined as follows (where $\Delta t = T/N$, N being a positive integer):

$$\operatorname*{Min}_{\mathbf{v}\in(L^2(\Gamma_0))^{N-1}} J_k^{\Delta t}(\mathbf{v}), \tag{2.61}$$

where $\mathbf{v} = \{v^n\}_{n=1}^{N-1}$ and

$$J_k^{\Delta t}(\mathbf{v}) = \frac{\Delta t}{2}\sum_{n=1}^{N-1} a_n \|v^n\|^2_{L^2(\Gamma_0)} + \frac{k}{2}\|y^N - y_T\|^2_{-1}; \tag{2.62}$$

in (2.62) we have $a_n = 1$ for $n = 1, 2, \ldots, N-2$, $a_{N-1} = 3/2$ and y^N obtained from $\mathbf{v}$ as follows:

$$y^0 = 0; \tag{2.63}$$

to obtain y^1 (respectively y^n, $n = 2, \ldots, N-1$) we solve the following *elliptic* problem

$$\frac{y^1 - y^0}{\Delta t} + A(\tfrac{2}{3}y^1 + \tfrac{1}{3}y^0) = 0 \text{ in } \Omega, \quad y^1 = v^1 \text{ on } \Gamma_0,\ y^1 = 0 \text{ on } \Gamma\backslash\Gamma_0 \tag{2.64}$$

(respectively

$$\frac{\frac{3}{2}y^n - 2y^{n-1} + \frac{1}{2}y^{n-2}}{\Delta t} + Ay^n = 0 \text{ in } \Omega, \quad y^n = v^n \text{ on } \Gamma_0, \; y^n = 0 \text{ on } \Gamma\backslash\Gamma_0); \tag{2.65}$$

finally y^N is defined via

$$\frac{2y^N - 3y^{N-1} + y^{N-2}}{\Delta t} + Ay^{N-1} = 0. \tag{2.66}$$

Problem (2.61) has a *unique* solution.

In order to discretize the *dual problem* (2.21) we look for the dual problem of the discrete control problem (2.61). The simplest way to derive the dual of problem (2.61) is to start from the *optimality condition*

$$\nabla J_k^{\Delta t}(\mathbf{u}^{\Delta t}) = \mathbf{0}, \tag{2.67}$$

where, in (2.67), $\mathbf{u}^{\Delta t} = \{u^n\}_{n=1}^{N-1}$ is *the* solution of the discrete control problem (2.61), and where $\nabla J_k^{\Delta t}$ denotes the gradient of the discrete cost function $J_k^{\Delta t}$. Suppose that the discrete control space $\mathcal{U}^{\Delta t} = (L^2(\Gamma_0))^{N-1}$ is equipped with the scalar product

$$(\mathbf{v}, \mathbf{w})_{\Delta t} = \Delta t \sum_{n=1}^{N-1} a_n \int_{\Gamma_0} v^n w^n \, \mathrm{d}\Gamma, \;\; \forall \mathbf{v}, \mathbf{w} \in \mathcal{U}^{\Delta t}; \tag{2.68}$$

then a tedious calculation will show that $\forall \mathbf{v}, \mathbf{w} \in \mathcal{U}^{\Delta t}$

$$\begin{aligned}(\nabla J_k^{\Delta t}(\mathbf{v}), \mathbf{w})_{\Delta t} \\ = \Delta t \sum_{n=1}^{N-2} \int_{\Gamma_0} \left(v^n - \frac{\partial p^n}{\partial n_{A^*}} \right) w^n \, \mathrm{d}\Gamma \\ + \frac{3}{2}\Delta t \int_{\Gamma_0} \left[v^{N-1} - \left(\frac{2}{3}\frac{\partial p^{N-1}}{\partial n_{A^*}} + \frac{1}{3}\frac{\partial p^N}{\partial n_{A^*}} \right) \right] w^{N-1} \, \mathrm{d}\Gamma,\end{aligned} \tag{2.69}$$

where, in (2.69), the *adjoint state vector* $\{p^n\}_{n=1}^N$ belongs to $(H_0^1(\Omega))^N$ and is obtained as follows.

First, compute p^N as the solution in $H_0^1(\Omega)$ of the elliptic problem

$$-\Delta p^N = k(y^N - y_T) \text{ in } \Omega, \quad p^N = 0 \text{ on } \Gamma, \tag{2.70}$$

then p^{N-1} (respectively p^n, $n = N-2, \ldots, 2, 1$) as the solution in $H_0^1(\Omega)$ of the elliptic problem

$$\frac{p^{N-1} - p^N}{\Delta t} + A^*(\tfrac{2}{3}p^{N-1} + \tfrac{1}{3}p^N) = 0 \text{ in } \Omega, \quad p^{N-1} = 0 \text{ on } \Gamma \tag{2.71}$$

(respectively

$$\frac{\frac{3}{2}p^n - 2p^{n+1} + \frac{1}{2}p^{n+2}}{\Delta t} + A^*p^n = 0 \text{ in } \Omega, \quad p^n = 0 \text{ on } \Gamma). \tag{2.72}$$

Combining (2.67) and (2.69) shows that the *optimal triple*

$$\{\mathbf{u}^{\Delta t}, \{y^n\}_{n=1}^N, \{p^n\}_{n=1}^N\}$$

is *characterized* by

$$u^n = \frac{\partial p^n}{\partial n_{A^*}}|_{\Gamma_0} \text{ if } n = 1, \ldots, N-2, \ u^{N-1} = \left(\frac{2}{3}\frac{\partial p^{N-1}}{\partial n_{A^*}} + \frac{1}{3}\frac{\partial p^N}{\partial n_{A^*}}\right)\Big|_{\Gamma_0}, \tag{2.73}$$

to be completed by (2.70)–(2.72) and by

$$y^0 = 0, \tag{2.74}$$

$$\frac{y^1 - y^0}{\Delta t} + A(\tfrac{2}{3}y^1 + \tfrac{1}{3}y^0) = 0 \text{ in } \Omega, \quad y^1 = u^1 \text{ on } \Gamma_0, \quad y^1 = 0 \text{ on } \Gamma\backslash\Gamma_0, \tag{2.75}$$

$$\frac{\frac{3}{2}y^n - 2y^{n-1} + \frac{1}{2}y^{n-2}}{\Delta t} + Ay^n = 0 \text{ in } \Omega, \quad y^n = u^n \text{ on } \Gamma_0, \ y^n = 0 \text{ on } \Gamma\backslash\Gamma_0 \tag{2.76}$$

if $n = 2, \ldots, N-1$,

$$\frac{2y^N - 3y^{N-1} + y^{N-2}}{\Delta t} + Ay^{N-1} = 0. \tag{2.77}$$

Following Section 2.2 we define $\Lambda^{\Delta t} \in \mathcal{L}(H_0^1(\Omega), \ H^{-1}(\Omega))$ by

$$\Lambda^{\Delta t}\hat{f} = -\hat{\varphi}^N, \ \forall \hat{f} \in H_0^1(\Omega), \tag{2.78}$$

where $\hat{\varphi}^N$ is obtained from $\hat{f}$ via the solution of the *discrete backward parabolic problem*

$$\hat{\psi}^N = \hat{f}, \tag{2.79}$$

$$\frac{\hat{\psi}^{N-1} - \hat{\psi}^N}{\Delta t} + A^*(\tfrac{2}{3}\hat{\psi}^{N-1} + \tfrac{1}{3}\hat{\psi}^N) = 0 \text{ in } \Omega, \quad \hat{\psi}^{N-1} = 0 \text{ on } \Gamma, \tag{2.80}$$

$$\frac{\frac{3}{2}\hat{\psi}^n - 2\hat{\psi}^{n+1} + \frac{1}{2}\hat{\psi}^{n+2}}{\Delta t} + A^*\hat{\psi}^n = 0 \text{ in } \Omega, \quad \hat{\psi}^n = 0 \text{ on } \Gamma \tag{2.81}$$

for $n = N-2, \ldots, 1$, and then of the *discrete forward parabolic problem*

$$\hat{\varphi}^0 = 0, \tag{2.82}$$

$$\frac{\hat{\varphi}^1 - \hat{\varphi}^0}{\Delta t} + A(\tfrac{2}{3}\hat{\varphi}^1 + \tfrac{1}{3}\hat{\varphi}^0) = 0 \text{ in } \Omega, \quad \hat{\varphi}^1 = \frac{\partial \hat{\psi}^1}{\partial n_{A^*}} \text{ on } \Gamma_0, \ \hat{\varphi}^1 = 0 \text{ on } \Gamma\backslash\Gamma_0, \tag{2.83}$$

$$\frac{\frac{3}{2}\hat{\varphi}^n - 2\hat{\varphi}^{n-1} + \frac{1}{2}\hat{\varphi}^{n-2}}{\Delta t} + A\hat{\varphi}^n = 0 \text{ in } \Omega, \quad \hat{\varphi}^n = \frac{\partial \hat{\psi}^n}{\partial n_{A^*}} \text{ on } \Gamma_0, \ \hat{\varphi}^n = 0 \text{ on } \Gamma\backslash\Gamma_0 \tag{2.84}$$

if $n = 2, \dots, N-2$,

$$\frac{\frac{3}{2}\hat{\varphi}^{N-1} - 2\hat{\varphi}^{N-2} + \frac{1}{2}\hat{\varphi}^{N-3}}{\Delta t} + A\hat{\varphi}^{N-1} = 0 \text{ in } \Omega,$$

$$\hat{\varphi}^{N-1} = \frac{2}{3}\frac{\partial\hat{\psi}^{N-1}}{\partial n_{A^*}} + \frac{1}{3}\frac{\partial\hat{\psi}^{N}}{\partial n_{A^*}} \text{ on } \Gamma_0, \ \hat{\varphi}^{N-1} = 0 \text{ on } \Gamma\backslash\Gamma_0, \tag{2.85}$$

$$\frac{2\hat{\varphi}^{N} - 3\hat{\varphi}^{N-1} + \hat{\varphi}^{N-2}}{\Delta t} + A\hat{\varphi}^{N-1} = 0. \tag{2.86}$$

We can show that (with obvious notation) we have, $\forall f_1, f_2 \in H_0^1(\Omega)$,

$$\begin{aligned}\langle\Lambda^{\Delta t}f_1, f_2\rangle \;=\; \Delta t\Bigg[\sum_{n=1}^{N-2}\int_{\Gamma_0}\frac{\partial\psi_1^n}{\partial n_{A^*}}\frac{\partial\psi_2^n}{\partial n_{A^*}}\,\mathrm{d}\Gamma + \frac{3}{2}\int_{\Gamma_0}\left(\frac{2}{3}\frac{\partial\psi_1^{N-1}}{\partial n_{A^*}} + \frac{1}{3}\frac{\partial\psi_1^{N}}{\partial n_{A^*}}\right)\\ \times\left(\frac{2}{3}\frac{\partial\psi_2^{N-1}}{\partial n_{A^*}} + \frac{1}{3}\frac{\partial\psi_2^{N}}{\partial n_{A^*}}\right)\mathrm{d}\Gamma\Bigg],\end{aligned} \tag{2.87}$$

where, in (2.87), $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

It follows from (2.87) that operator $\Lambda^{\Delta t}$ is *self-adjoint* and *positive semi-definite* over $H_0^1(\Omega)$.

Back to the optimality system (2.70)–(2.77), let us denote by $f^{\Delta t}$ the function p^N; it follows then from the definition of $\Lambda^{\Delta t}$ that (2.70) can be reformulated as

$$-k^{-1}\Delta f^{\Delta t} + \Lambda^{\Delta t}f^{\Delta t} = -y_T, \tag{2.88}$$

which is precisely the dual problem we have been looking for. The full space/time discretization of problems (2.12) and (2.21) will be discussed in the following.

2.4.3. Full space/time discretization of problems (2.12) and (2.21)

The *full discretization* of control problems, related to (2.12) and (2.21), has been already discussed in Sections 1.8.4 and 1.10.6. Despite many similarities, the *boundary control* problems discussed here are substantially more complicated to fully discretize than the above distributed and pointwise control problems. The main reason for this increased complexity arises from the fact that we still intend to employ *low-order finite element* approximations – as in Sections 1.8 and 1.10 – to space discretize the parabolic state problem (2.1)–(2.3) and the corresponding adjoint system (2.15). With such a choice the 'obvious' approximations of $\partial/\partial n_{A^*}|_{\Gamma_0}$ will be fairly inaccurate. In order to obtain second-order accurate approximations of $\partial/\partial n_{A^*}|_{\Gamma_0}$, we shall rely on a discrete Green's formula, following a strategy which has been successfully used in, e.g., Glowinski *et al.* (1990), Glowinski (1992a) (for the boundary control of the *wave equation*) and Carthel *et al.* (1994) (for the boundary control of the *heat equation*).

We suppose for simplicity that Ω is a *bounded* polygonal domain of $\mathbb{R}^2$. We introduce then, as in Sections 1.8.4. and 1.10.6, a *triangulation* $\mathcal{T}_h$ of Ω (h: largest length of the edges of the triangles of $\mathcal{T}_h$). Next, we approximate $H^1(\Omega), L^2(\Omega)$ and $H_0^1(\Omega)$ by

$$H_h^1 = \{z_h | z_h \in C^0(\bar{\Omega}), z_h|_T \in P_1, \forall T \in \mathcal{T}_h\}, \tag{2.89}$$

$$H_{0h}^1 = \{z_h | z_h \in H_h^1, z_h = 0 \text{ on } \Gamma\} (= H_0^1(\Omega) \cap H_h^1), \tag{2.90}$$

respectively (with, as usual, P_1 the space of polynomials in x_1, x_2 of degree ≤ 1). Another important finite element space is

$$V_{0h} = \{z_h \mid z_h \in H_h^1, z_h = 0 \text{ on } \Gamma \backslash \Gamma_0\}; \tag{2.91}$$

if $\int_{\Gamma\backslash\Gamma_0} \mathrm{d}\Gamma > 0$ we shall assume that those boundary points at the interface of Γ_0 and $\Gamma\backslash\Gamma_0$ are vertices of $\mathcal{T}_h$. Finally, the role of $L^2(\Gamma_0)$ will be played by the space $M_h(\subset V_{0h})$ defined as follows:

$$M_h \oplus H_{0h}^1 = V_{0h}, \ \mu_h \in M_h \Rightarrow \mu_h \,|_T = 0, \ \forall T \in \mathcal{T}_h, \quad \text{such as } \ \partial T \cap \Gamma = \emptyset. \tag{2.92}$$

Space M_h is clearly *isomorphic* to the boundary space consisting of the traces on Γ of those functions belonging to V_{0h}; also, $\dim(M_h)$ is equal to the number of $\mathcal{T}_h$ boundary vertices interior to Γ_0 and the following bilinear form

$$\{\lambda_h, \mu_h\} \to \int_{\Gamma_0} \lambda_h \mu_h \,\mathrm{d}\Gamma$$

defines a scalar product on M_h.

Since the full space/time discretization of problems (2.12) and (2.21) will rely on *variational* techniques, it is convenient to introduce the bilinear form $a: \ H^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$a(y, z) = \langle Ay, z \rangle, \ \forall y \in H^1(\Omega), \ \forall z \in H_0^1(\Omega), \tag{2.93}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Assuming that the coefficients of the second-order elliptic operator A are sufficiently smooth we also have

$$a(y, z) = \int_\Omega (A^* z) y \,\mathrm{d}x + \int_\Gamma \frac{\partial z}{\partial n_{A^*}} y \,\mathrm{d}\Gamma, \ \forall y \in H^1(\Omega), \ \forall z \in H_0^1(\Omega) \cap H^2(\Omega), \tag{2.94}$$

which is definitely a generalization of the well-known *Green's formula*

$$\int_\Omega \nabla y \cdot \nabla z \,\mathrm{d}x = -\int_\Omega \Delta z y \,\mathrm{d}x + \int_\Gamma \frac{\partial z}{\partial n} y \,\mathrm{d}\Gamma, \ \forall y \in H^1(\Omega), \ \forall z \in H_0^1(\Omega) \cap H^2(\Omega).$$

Following Section 2.4.2 we approximate the control problem (2.12) by

$$\underset{\mathbf{v} \in \mathcal{U}_h^{\Delta t}}{\operatorname{Min}} J_h^{\Delta t}(\mathbf{v}), \tag{2.95}$$

where, in (2.95), we have $\mathcal{U}_h^{\Delta t} = (M_h)^{N-1}$, $\mathbf{v} = \{v^n\}_{n=1}^{N-1}$ and

$$J_h^{\Delta t}(\mathbf{v}) = \frac{\Delta t}{2}\sum_{n=1}^{N-1} a_n \int_\Omega |v^n|^2 \, d\Gamma + \frac{k}{2}\int_\Gamma |\nabla \phi^N|^2 \, dx, \tag{2.96}$$

with, in (2.96), ϕ^N obtained from $\mathbf{v}$ via the solution of the following well-posed *discrete parabolic* and *elliptic* problems

Parabolic problem.

$$y^0 = 0; \tag{2.97}$$

compute y^1 from

$$\begin{cases} y^1 \in V_{0h}, \ y^1 = v^1 \text{ on } \Gamma_0, \\ \displaystyle\int_\Omega \frac{y^1 - y^0}{\Delta t} z \, dx + a(\tfrac{2}{3}y^1 + \tfrac{1}{3}y^0, z) = 0, \ \forall z \in H_{0h}^1, \end{cases} \tag{2.98}$$

then y^n from

$$\begin{cases} y^n \in V_{0h}, \ y^n = v^n \text{ on } \Gamma_0, \\ \displaystyle\int_\Omega \frac{\frac{3}{2}y^n - 2y^{n-1} + \frac{1}{2}y^{n-2}}{\Delta t} z \, dx + a(y^n, z) = 0, \ \forall z \in H_{0h}^1 \end{cases} \tag{2.99}$$

for $n = 2, \ldots, N-1$, and y^N from

$$\begin{cases} y^N \in H_{0h}^1, \\ \displaystyle\int_\Omega \frac{2y^N - 3y^{N-1} + y^{N-2}}{\Delta t} z \, dx + a(y^{N-1}, z) = 0, \ \forall z \in H_{0h}^1. \end{cases} \tag{2.100}$$

Elliptic problem.

$$\begin{cases} \phi^N \in H_{0h}^1, \\ \displaystyle\int_\Omega \nabla \phi^N \cdot \nabla z \, dx = \int_\Omega (y^N - y_T) z \, dx, \ \forall z \in H_{0h}^1. \end{cases} \tag{2.101}$$

We then have the following

Proposition 2.2 *The discrete control problem (2.95) has a unique solution $\mathbf{u}_h^{\Delta t} = \{u^n\}_{n=1}^{N-1}$. If we denote by $\mathbf{y}_h^{\Delta t} = \{y^n\}_{n=0}^{N}$ the solution of (2.97)–(2.100) associated with $\mathbf{v} = \mathbf{u}_h^{\Delta t}$, the optimal pair $\{\mathbf{u}_h^{\Delta t}, \mathbf{y}_h^{\Delta t}\}$ is characterized by the existence of $\mathbf{p}_h^{\Delta t} = \{p^n\}_{n=1}^{N} \in (H_{0h}^1)^N$ such that*

$$\begin{cases} p^N \in H_{0h}^1, \\ \displaystyle\int_\Omega \nabla p^N \cdot \nabla z \, dx = k\left[\int_\Omega y^N z \, dx - \langle y_T, z\rangle\right], \ \forall z \in H_{0h}^1, \end{cases} \tag{2.102}$$

($\langle \cdot, \cdot \rangle$: duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$),

$$\begin{cases} p^{N-1} \in H_{0h}^1, \\ \displaystyle\int_\Omega \frac{p^{N-1} - p^N}{\Delta t} z \, dx + a(z, \tfrac{2}{3}p^{N-1} + \tfrac{1}{3}p^N) = 0, \ \forall z \in H_{0h}^1, \end{cases} \tag{2.103}$$

$$\begin{cases} p^n \in H^1_{0h}, \\ \int_\Omega \frac{\frac{3}{2}p^n - 2p^{n+1} + \frac{1}{2}p^{n+2}}{\Delta t} z \, dx + a(z, p^n) = 0, \ \forall z \in H^1_{0h} \end{cases} \tag{2.104}$$

for $n = N-2, \dots, 1$, and also

$$\begin{cases} u^n \in M_h, \\ \int_{\Gamma_0} u^n \mu \, d\Gamma = \int_\Omega \frac{\frac{3}{2}p^n - 2p^{n+1} + \frac{1}{2}p^{n+2}}{\Delta t} \mu \, dx + a(\mu, p^n), \ \forall \mu \in M_h \end{cases} \tag{2.105}$$

if $n = 1, 2, \dots, N-2$, and finally

$$\begin{cases} u^{N-1} \in M_h, \\ \int_\Gamma u^{N-1} \mu \, d\Gamma = \int_\Omega \frac{p^{N-1} - p^N}{\Delta t} \mu \, dx + a(\mu, \frac{2}{3}p^{N-1} + \frac{1}{3}p^N), \ \forall \mu \in M_h \end{cases} \tag{2.106}$$

Proof. The *existence* and *uniqueness* properties are obvious. Concerning now the relations characterizing the optimal pair $\{\mathbf{u}_h^{\Delta t}, \mathbf{y}_h^{\Delta t}\}$ they follow from the *optimality condition*

$$\nabla J_h^{\Delta t}(\mathbf{u}_h^{\Delta t}) = \mathbf{0}, \tag{2.107}$$

where $\nabla J_h^{\Delta t}$ is the gradient of the functional $J_h^{\Delta t}$. Indeed, if we use

$$(\mathbf{v}, \mathbf{w})_{\Delta t} = \Delta t \sum_{n=1}^{N-1} a_n \int_{\Gamma_0} v^n w^n \, d\Gamma$$

as the scalar product over $\mathcal{U}_h^{\Delta t}$, it can be shown that, $\forall \mathbf{v}, \mathbf{w} \in \mathcal{U}_h^{\Delta t}$, we have

$$\begin{aligned} &(\nabla J_h^{\Delta t}(\mathbf{v}), \mathbf{w})_{\Delta t} \\ &= \Delta t \textstyle\sum_{n=1}^{N-2} \left[\int_{\Gamma_0} v^n w^n \, d\Gamma - \int_\Omega \frac{\frac{3}{2}p^n - 2p^{n+1} + \frac{1}{2}p^{n+2}}{\Delta t} w^n \, dx - a(w^n, p^n) \right] \\ &\qquad + \tfrac{3}{2}\Delta t \Big[\int_{\Gamma_0} v^{N-1} w^{N-1} \, d\Gamma - \int_\Omega \tfrac{p^{N-1} - p^N}{\Delta t} w^{N-1} \, dx \\ &\qquad\qquad - a(w^{N-1}, \tfrac{2}{3}p^{N-1} + \tfrac{1}{3}p^N) \Big], \end{aligned} \tag{2.108}$$

where, in (2.108), $\{p^n\}_{n=1}^{N-1}$ is obtained from $\mathbf{v} = \{v^n\}_{n=1}^{N-1}$ via the solution of the discrete parabolic and elliptic problems (2.97)–(2.100), (2.102) and (2.103), (2.104). Relations (2.107) and (2.108) clearly imply (2.102)–(2.106).

Remark 2.4 Relations (2.105), (2.106) are not that mysterious. For the continuous problem (2.12), we know (see Section 2.2) that the optimal control u satisfies

$$u = \frac{\partial p}{\partial n_{A^*}} \text{ on } \Sigma_0, \tag{2.109}$$

where p is the solution of the corresponding adjoint system (2.15). We have thus $\partial p/\partial t = A^*p$, which combined with Green's formula (2.94), implies that a.e. on $(0,T)$ we have

$$\int_{\Gamma_0} u\mu \, d\Gamma = -\int_\Omega \frac{\partial p}{\partial t}\mu \, dx + a(\mu, p), \ \forall \mu \in H^1(\Omega), \ \ \mu = 0 \text{ on } \Gamma\backslash\Gamma_0. \quad (2.110)$$

Relations (2.105), (2.106) are clearly discrete analogues of (2.110).

To obtain the *fully discrete* analogue of the *dual problems* (2.21) and (2.88) we introduce $\Lambda_h^{\Delta t} \in \mathcal{L}(H_{0h}^1, H_{0h}^1)$ defined as follows

$$\Lambda_h^{\Delta t}\hat{f} = -\hat{\varphi}^N, \ \forall \hat{f} \in H_{0h}^1, \quad (2.111)$$

where $\hat{\varphi}^N$ is obtained from $\hat{f}$ via the solution of the *fully discrete backward parabolic problem*

$$\hat{\psi}^N = \hat{f}, \quad (2.112)$$

$$\begin{cases} \hat{\psi}^{N-1} \in H_{0h}^1, \\ \displaystyle\int_\Omega \frac{\hat{\psi}^{N-1} - \hat{\psi}^N}{\Delta t} z \, dx + a(z, \tfrac{2}{3}\hat{\psi}^{N-1} + \tfrac{1}{3}\hat{\psi}^N) = 0, \ \forall z \in H_{0h}^1, \end{cases} \quad (2.113)$$

$$\begin{cases} \hat{\psi}^n \in H_{0h}^1, \\ \displaystyle\int_\Omega \frac{\tfrac{3}{2}\hat{\psi}^n - 2\hat{\psi}^{n+1} + \tfrac{1}{2}\hat{\psi}^{n+2}}{\Delta t} z \, dx + a(z, \hat{\psi}^n) = 0, \ \forall z \in H_{0h}^1 \end{cases} \quad (2.114)$$

for $n = N-2, \dots, 1$, and then of the *fully discrete forward parabolic problem*

$$\hat{\varphi}^0 = 0, \quad (2.115)$$

$$\begin{cases} \hat{\varphi}^1 \in V_{0h}, \ \hat{\varphi}^1 = \hat{u}^1 \text{ on } \Gamma_0, \\ \displaystyle\int_\Omega \frac{\hat{\varphi}^1 - \hat{\varphi}^0}{\Delta t} z \, dx + a(\tfrac{2}{3}\hat{\varphi}^1 + \tfrac{1}{3}\hat{\varphi}^0, z) = 0, \ \forall z \in H_{0h}^1, \end{cases} \quad (2.116)$$

$$\begin{cases} \hat{\varphi}^n \in V_{0h}, \ \hat{\varphi}^n = \hat{u}^n \text{ on } \Gamma_0, \\ \displaystyle\int_\Omega \frac{\tfrac{3}{2}\hat{\varphi}^n - 2\hat{\varphi}^{n-1} + \tfrac{1}{2}\hat{\varphi}^{n-2}}{\Delta t} z \, dx + a(\hat{\varphi}^n, z) = 0, \ \forall z \in H_{0h}^1 \end{cases} \quad (2.117)$$

for $n = 2, \dots, N-1$, and finally

$$\begin{cases} \hat{\varphi}^N \in H_{0h}^1, \\ \displaystyle\int_\Omega \frac{2\hat{\varphi}^N - 3\hat{\varphi}^{N-1} + \hat{\varphi}^{N-2}}{\Delta t} z \, dx + a(\hat{\varphi}^{N-1}, z) = 0, \ \forall z \in H_{0h}^1; \end{cases} \quad (2.118)$$

in (2.116), (2.117) the vector $\{\hat{u}^n\}_{n=1}^{N-1}$ is defined from $\{\hat{\psi}^n\}_{n=1}^N$ as follows

$$\begin{cases} \hat{u}^n \in M_h, \\ \displaystyle\int_{\Gamma_0} \hat{u}^n \mu \, d\Gamma = \int_\Omega \frac{\tfrac{3}{2}\hat{\psi}^n - 2\hat{\psi}^{n+1} + \tfrac{1}{2}\hat{\psi}^{n+2}}{\Delta t} \mu \, dx + a(\mu, \hat{\psi}^n), \ \forall \mu \in M_h \end{cases} \quad (2.119)$$

if $n = 1, 2, \ldots, N-2$, and

$$\begin{cases} \hat{u}^{N-1} \in M_h, \\ \int_{\Gamma_0} \hat{u}^{N-1} \mu \, d\Gamma = \int_\Omega \frac{\hat{\psi}^{N-1} - \hat{\psi}^N}{\Delta t} \mu \, dx \\ \qquad + a(\mu, \tfrac{2}{3}\hat{\psi}^{N-1} + \tfrac{1}{3}\hat{\psi}^N), \ \forall \mu \in M_h. \end{cases} \tag{2.120}$$

We can show that

$$\int_\Omega (\Lambda_h^{\Delta t} f_1) f_2 \, dx = \Delta t \sum_{n=1}^{N-1} a_n \int_{\Gamma_0} u_1^n u_2^n \, d\Gamma, \ \forall f_1, f_2 \in H_{0h}^1, \tag{2.121}$$

where, in (2.121), $\{u_i^n\}_{n=1}^{N-1}$, $i = 1, 2$ is obtained from f_i via (2.112)–(2.114), (2.119), (2.120).

It follows from (2.121) that operator $\Lambda_h^{\Delta t}$ is *symmetric* and *positive semi-definite* over H_{0h}^1. □

Let us consider now the *optimal triple* $\{\mathbf{u}_h^{\Delta t}, \mathbf{y}_h^{\Delta t}, \mathbf{p}_h^{\Delta t}\}$ and define $f_h^{\Delta t} \in H_h^1$ by

$$f_h^{\Delta t} = p^N. \tag{2.122}$$

It follows then from Proposition 2.2 and from the definition of $\Lambda_h^{\Delta t}$ that

$$\Lambda_h^{\Delta t} f_h^{\Delta t} = -y^N. \tag{2.123}$$

Combining (2.123) with (2.102) we obtain

$$\begin{cases} f_h^{\Delta t} \in H_{0h}^1, \\ k^{-1} \int_\Omega \nabla f_h^{\Delta t} \cdot \nabla z \, dx + \int_\Omega \Lambda_h^{\Delta t} f_h^{\Delta t} z \, dx = -\langle y_T, z \rangle, \ \forall z \in H_{0h}^1. \end{cases} \tag{2.124}$$

Problem (2.124) is precisely the *fully discrete dual problem* we were looking for. From the properties of $\Lambda_h^{\Delta t}$ (*symmetry* and *semi-positiveness*), problem (2.124) can be solved by a *conjugate gradient* algorithm operating in H_{0h}^1 (a fully discrete analogue of algorithm (2.42)–(2.54)); we shall describe this algorithm in Section 2.5.

Remark 2.5 From a practical point of view, it makes sense to use the *trapezoidal rule* to (approximately) compute the various $L^2(\Omega)$ and $L^2(\Gamma_0)$ scalar products occurring in the definition of the approximate control problem (2.95), and of its dual problem (2.124). If this approach is retained, the corresponding operator $\Lambda_h^{\Delta t}$ has the same basic properties as that defined by (2.111), namely *symmetry* and *semi-positiveness*, implying that the corresponding variant of problem (2.121) can also be solved by a *conjugate gradient algorithm* operating in the space H_{0h}^1.

2.5. Dirichlet control (V): Iterative solution of the fully discrete dual problem (2.124)

We have described in Section 2.3 a *conjugate gradient algorithm* for solving the control problem (2.12), either *directly* (by algorithm (2.27)–(2.41); see Section 2.3.1) or via the solution of the *dual problem* (2.21) (by algorithm (2.42)–(2.54); see Section 2.3.2). Since the numerical results presented in the following section were obtained via the solution of the dual problem we shall focus our discussion on the *iterative solution* of the *fully discrete* approximation of problem (2.21) (i.e. problem (2.124)). From the properties of $\Lambda_h^{\Delta t}$ problem (2.124) can be solved by a *conjugate gradient algorithm* operating in the finite dimensional space H_{0h}^1. From Sections 1.8.2 and 2.3.2 this algorithm takes the following form:

$$f_0 \text{ is given in } H_{0h}^1; \tag{2.125}$$

set

$$p_0^N = f_0 \tag{2.126}$$

and solve first

$$\begin{cases} p_0^{N-1} \in H_{0h}^1, \\ \displaystyle\int_\Omega \frac{p_0^{N-1} - p_0^N}{\Delta t} z\,dx + a(z, \tfrac{2}{3}p_0^{N-1} + \tfrac{1}{3}p_0^N) = 0, \ \forall z \in H_{0h}^1 \end{cases} \tag{2.127}$$

and

$$\begin{cases} u_0^{N-1} \in M_h, \\ \displaystyle\int_{\Gamma_0} u_0^{N-1}\mu\,d\Gamma = \int_\Omega \frac{p_0^{N-1} - p_0^N}{\Delta t}\mu\,dx + a(\mu, \tfrac{2}{3}p_0^{N-1} + \tfrac{1}{3}p_0^N), \ \forall \mu \in M_h, \end{cases} \tag{2.128}$$

and then for $n = N-2, \ldots, 1$

$$\begin{cases} p_0^n \in H_{0h}^1, \\ \displaystyle\int_\Omega \frac{\tfrac{3}{2}p_0^n - 2p_0^{n+1} + \tfrac{1}{2}p_0^{n+2}}{\Delta t} z\,dx + a(z, p_0^n) = 0, \ \forall z \in H_{0h}^1, \end{cases} \tag{2.129}$$

$$\begin{cases} u_0^n \in M_h, \\ \displaystyle\int_{\Gamma_0} u_0^n\mu\,d\Gamma = \int_\Omega \frac{\tfrac{3}{2}p_0^n - 2p_0^{n+1} + \tfrac{1}{2}p_0^{n+2}}{\Delta t}\mu\,dx + a(\mu, p_0^n), \ \forall \mu \in M_h. \end{cases} \tag{2.130}$$

Solve next the following fully discrete forward parabolic problem

$$y_0^0 = 0, \tag{2.131}$$

$$\begin{cases} y_0^1 \in V_{0h}, \ y_0^1 = u_0^1 \text{ on } \Gamma_0, \\ \displaystyle\int_\Omega \frac{y_0^1 - y_0^0}{\Delta t} z\,dx + a(\tfrac{2}{3}y_0^1 + \tfrac{1}{3}y_0^0, z) = 0, \ \forall z \in H_{0h}^1, \end{cases} \tag{2.132}$$

$$\begin{cases} y_0^n \in V_{0h}, \; y_0^n = u_0^n \text{ on } \Gamma_0, \\ \displaystyle\int_\Omega \frac{\frac{3}{2}y_0^n - 2y_0^{n-1} + \frac{1}{2}y_0^{n-2}}{\Delta t} z \, \mathrm{d}x + a(y_0^n, z) = 0, \; \forall z \in H_{0h}^1 \end{cases} \tag{2.133}$$

for $n = 2, \ldots, N-1$ *and finally*

$$\begin{cases} y_0^N \in H_{0h}^1, \\ \displaystyle\int_\Omega \frac{2y_0^N - 3y_0^{N-1} + y_0^{N-2}}{\Delta t} z \, \mathrm{d}x + a(y_0^{N-1}, z) = 0, \; \forall z \in H_{0h}^1. \end{cases} \tag{2.134}$$

Solve now

$$\begin{cases} g_0 \in H_{0h}^1, \\ \displaystyle\int_\Omega \nabla g_0 \cdot \nabla z \, \mathrm{d}x = k^{-1} \int_\Omega \nabla f_0 \cdot \nabla z \, \mathrm{d}x + \langle y_T, z \rangle - \int_\Omega y_0^N z \, \mathrm{d}x, \; \forall z \in H_{0h}^1 \end{cases} \tag{2.135}$$

and set

$$w_0 = g_0. \tag{2.136}$$

Then for $m \geq 0$, *assuming that* f_m, g_m, w_m *are known, compute* f_{m+1}, g_{m+1}, w_{m+1} *as follows*:
Take

$$\bar{p}_m^N = w_m \tag{2.137}$$

and solve

$$\begin{cases} \bar{p}_m^{N-1} \in H_{0h}^1, \\ \displaystyle\int_\Omega \frac{\bar{p}_m^{N-1} - \bar{p}_m^N}{\Delta t} z \, \mathrm{d}x + a(z, \tfrac{2}{3}\bar{p}_m^{N-1} + \tfrac{1}{3}\bar{p}_m^N) = 0, \; \forall z \in H_{0h}^1 \end{cases} \tag{2.138}$$

and

$$\begin{cases} \bar{u}_m^{N-1} \in M_h, \\ \displaystyle\int_{\Gamma_0} \bar{u}_m^{N-1} \mu \, \mathrm{d}\Gamma = \int_\Omega \frac{\bar{p}_m^{N-1} - \bar{p}_m^N}{\Delta t} \mu \, \mathrm{d}x + a(\mu, \tfrac{2}{3}\bar{p}_m^{N-1} + \tfrac{1}{3}\bar{p}_m^N) = 0, \, \forall \mu \in M_h, \end{cases} \tag{2.139}$$

and then for $n = N-2, \ldots, 1$

$$\begin{cases} \bar{p}_m^n \in H_{0h}^1, \\ \displaystyle\int_\Omega \frac{\frac{3}{2}\bar{p}_m^n - 2\bar{p}_m^{n+1} + \frac{1}{2}\bar{p}_m^{n+2}}{\Delta t} z \, \mathrm{d}x + a(z, \bar{p}_m^n) = 0, \; \forall z \in H_{0h}^1, \end{cases} \tag{2.140}$$

$$\begin{cases} \bar{u}_m^n \in M_h, \\ \displaystyle\int_{\Gamma_0} \bar{u}_m^n \mu \, \mathrm{d}\Gamma = \int_\Omega \frac{\frac{3}{2}\bar{p}_m^n - 2\bar{p}_m^{n+1} + \frac{1}{2}\bar{p}_m^{n+2}}{\Delta t} \mu \, \mathrm{d}x + a(\mu, \bar{p}_m^n), \; \forall \, \mu \in M_h. \end{cases} \tag{2.141}$$

Solve next the following discrete forward parabolic problem

$$\bar{y}_m^0 = 0, \tag{2.142}$$

$$\begin{cases} \bar{y}_m^1 \in V_{0h},\ \bar{y}_m^1 = \bar{u}_m^1 \text{ on } \Gamma_0, \\ \displaystyle\int_\Omega \frac{\bar{y}_m^1 - \bar{y}_m^0}{\Delta t} z\,\mathrm{d}x + a(\tfrac{2}{3}\bar{y}_m^1 + \tfrac{1}{3}\bar{y}_m^0, z) = 0,\ \forall z \in H_{0h}^1, \end{cases} \tag{2.143}$$

$$\begin{cases} \bar{y}_m^n \in V_{0h},\ \bar{y}_m^n = \bar{u}_m^n \text{ on } \Gamma_0, \\ \displaystyle\int_\Omega \frac{\frac{3}{2}\bar{y}_m^n - 2\bar{y}_m^{n-1} + \frac{1}{2}\bar{y}_m^{n-2}}{\Delta t} z\,\mathrm{d}x + a(\bar{y}_m^n, z) = 0,\ \forall z \in H_{0h}^1 \end{cases} \tag{2.144}$$

for $n = 2, \ldots, N-1$, *and finally*

$$\begin{cases} \bar{y}_m^N \in H_{0h}^1, \\ \displaystyle\int_\Omega (2\bar{y}_m^N - 3\bar{y}_m^{N-1} + \bar{y}_m^{N-2}) z\,\mathrm{d}x + \Delta t a(\bar{y}_m^{N-1}, z) = 0,\ \forall z \in H_{0h}^1. \end{cases} \tag{2.145}$$

Solve now

$$\begin{cases} \bar{g}_m \in H_{0h}^1, \\ \displaystyle\int_\Omega \nabla \bar{g}_m \cdot \nabla z\,\mathrm{d}x = k^{-1} \int_\Omega \nabla w_m \cdot \nabla z\,\mathrm{d}x - \int_\Omega \bar{y}_m^N z\,\mathrm{d}x,\ \forall z \in H_{0h}^1, \end{cases} \tag{2.146}$$

and compute

$$\rho_m = \int_\Omega |\nabla g_m|^2\,\mathrm{d}x \Big/ \int_\Omega \nabla \bar{g}_m \cdot \nabla w_m\,\mathrm{d}x, \tag{2.147}$$

$$f_{m+1} = f_m - \rho_m w_m, \tag{2.148}$$

$$g_{m+1} = g_m - \rho_m \bar{g}_m. \tag{2.149}$$

If $\|g_{m+1}\|_{H_0^1(\Omega)} / \|g_0\|_{H_0^1(\Omega)} \le \epsilon$ *take* $f_h^{\Delta t} = f_{m+1}$ *and solve* (2.112)–(2.120) *with* $\hat{f} = f_h^{\Delta t}$ *to obtain* $\mathbf{u}_h^{\Delta t} = \{u^n\}_{n=1}^{N-1}$; *if the above stopping test is not satisfied, compute*

$$\gamma_m = \int_\Omega |\nabla g_{m+1}|^2\,\mathrm{d}x \Big/ \int_\Omega |\nabla g_m|^2\,\mathrm{d}x, \tag{2.150}$$

and then

$$w_{m+1} = g_{m+1} + \gamma_m w_m. \tag{2.151}$$

Do $m = m + 1$ *and go to* (2.137).

Remark 2.6 Algorithm (2.125)–(2.151) may seem complicated (27 instructions); in fact it is quite easy to implement since it essentially requires a *fast elliptic solver*; for the calculations presented in Section 2.6 we have been using a *multigrid* based elliptic solver (see, e.g., Hackbush (1985), Yserentant (1993) and the references therein for a thorough discussion of the solution of discrete elliptic problems by multigrid methods).

Remark 2.7 If h and Δt are sufficiently small Remarks 2.2 and 2.3 still hold for algorithm (2.125)–(2.151).

2.6. Dirichlet control (VI): Numerical experiments

2.6.1. First test problem

The first test problem is one for which the *exact controllability property holds*; indeed, to construct more easily a test problem whose exact solution is known we have taken a *nonzero source term* in the right-hand side of the state equation (2.1), obtaining thus

$$\frac{\partial y}{\partial t} + Ay = s \text{ in } Q, \tag{2.1$'$}$$

and also replaced the initial condition (2.3) by

$$y(0) = y_0, \tag{2.3$'$}$$

with $y_0 \neq 0$. For these numerical experiments we have taken $\Omega = (0,1) \times (0,1)$, $\Gamma_0 = \Gamma$, $T = 1$ and $A = -\nu\Delta$, with $\nu > 0$ ($(2.1)'$ is therefore a *heat equation*); the *source term* s, the *initial value* y_0 and the *target function* y_T are defined by

$$s(x_1, x_2, t) = 3\pi^3 \nu \mathrm{e}^{2\pi^2 \nu t}(\sin \pi x_1 + \sin \pi x_2), \tag{2.152}$$

$$y_0(x_1, x_2) = \pi(\sin \pi x_1 + \sin \pi x_2), \tag{2.153}$$

$$y_T(x_1, x_2) = \pi \mathrm{e}^{2\pi^2 \nu}(\sin \pi x_1 + \sin \pi x_2), \tag{2.154}$$

respectively.

With these data the (unique) solution u of the optimal control problem

$$\operatorname*{Min}_{v \in \mathcal{U}_f} J(v) \tag{2.155}$$

(with

$$\begin{aligned} J(v) &= \frac{1}{2}\int_\Sigma |v|^2 \,\mathrm{d}\Gamma \,\mathrm{d}t, \\ \mathcal{U}_f &= \{v \mid v \in L^2(\Sigma), \text{ the pair } \{v, y\} \\ &\quad \text{satisfies } (2.1)', (2.2), (2.3)' \text{ and } y(T) = y_T\}) \end{aligned}$$

is given by

$$\begin{cases} u(x_1, x_2, t) = \pi \mathrm{e}^{2\pi^2 \nu t} \sin \pi x_1 \text{ if } 0 < x_1 < 1 \text{ and } x_2 = 0 \text{ or } 1, \\ u(x_1, x_2, t) = \pi \mathrm{e}^{2\pi^2 \nu t} \sin \pi x_2 \text{ if } 0 < x_2 < 1 \text{ and } x_1 = 0 \text{ or } 1, \end{cases} \tag{2.156}$$

the corresponding function y being defined by

$$y(x_1, x_2, t) = \pi \mathrm{e}^{2\pi^2 \nu t}(\sin \pi x_1 + \sin \pi x_2). \tag{2.157}$$

Concerning now the *dual problem* of (2.155) we can easily show that it is defined by

$$\Lambda f = Y_0(T) - y_T, \tag{2.158}$$

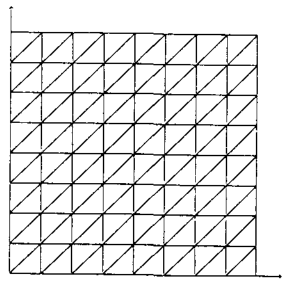


Fig. 1. A regular triangulation of $(0,1) \times (0,1)$.

where operator Λ is still defined by (2.17)–(2.19) and where the function Y_0 is the solution of

$$\frac{\partial Y_0}{\partial t} + AY_0 = s \text{ on } Q, \quad Y_0(0) = y_0, \ Y_0 = 0 \text{ on } \Sigma. \tag{2.159}$$

Since the data have been chosen so that we have exact controllability, the dual problem (2.158) has a unique solution which, in the particular case discussed here, is given by

$$f(x_1, x_2) = \pi e^{2\pi^2 \nu} \sin \pi x_1 \sin \pi x_2. \tag{2.160}$$

To approximate problem (2.158) (and therefore problem (2.155)) we have used the method described in Sections 2.3 to 2.5, namely: *time discretization* by a *second-order accurate scheme, space discretization* by *finite element methods* (using *regular* triangulations $\mathcal{T}_h$ like the one in Figure 1) and *iterative solution* by a trivial variant of algorithm (2.125)–(2.151) (with $k = +\infty$) with $\epsilon = 10^{-4}$ for the stopping criterium.

The above solution methodology has been tested for various values of h and Δt; for all of them, we have taken $\nu = 1/2\pi^2 (= 5.066059\ldots \times 10^{-2})$. On Table 1 we have summarized the results which have been obtained (we have used a $*$ to indicate a *computed* quantity). All the calculations have been done with $f_0 = 0$ as initializer for the conjugate gradient algorithms.

The results presented in Table 1 deserve some comments:

1. The convergence of the conjugate gradient algorithm is fairly fast if we keep in mind that the solution $f_h^{\Delta t}$ of the discrete problem which has been solved can be viewed as a vector with $(31)^2 = 961$ components if $h = \Delta t = 1/32$ (respectively $(63)^2 = 3969$ components if $h = \Delta t = 1/64$).
2. The target function y_T has been reached within a good accuracy, similar comments holding for the approximation of the optimal control u and of the solution f to the dual problem (2.158).
3. For information, we have $\|u/\|_{L^2(\Sigma)} = \pi\sqrt{e^2 - 1} = 7.94087251\ldots$ and $\|f\|_{H_0^1(\Omega)} = \pi e/\sqrt{2} = 6.03850398\ldots$.

Table 1. *Summary of numerical results.*

	$h = \Delta t = \frac{1}{32}$	$h = \Delta t = \frac{1}{64}$
Number of iterations	10	11
$\frac{\Vert y_T^* - y_T \Vert_{-1}}{\Vert y_T \Vert_{-1}}$	2.24×10^{-5}	1.78×10^{-5}
$\Vert u^* \Vert_{L^2(\Sigma)}$	7.791	7.863
$\frac{\Vert u^* - u \Vert_{L^2(\Sigma)}}{\Vert u \Vert_{L^2(\Sigma)}}$	2.50×10^{-3}	1.21×10^{-3}
$\Vert f^* \Vert_{H_0^1(\Omega)}$	6.07	6.041
$\frac{\Vert f^* - f \Vert_{H_0^1(\Omega)}}{\Vert f \Vert_{H_0^1(\Omega)}}$	2.44×10^{-2}	2.85×10^{-2}
$\frac{\Vert f^* - f \Vert_{L^2(\Omega)}}{\Vert f \Vert_{L^2(\Omega)}}$	6.53×10^{-3}	7.02×10^{-3}

On Figures 2 and 3 we have compared $y_T(x_1, 0.5)$ (...) and $y_T^*(x_1, 0.5)$ (—) for $x_1 \in (0,1)$ and $h = \Delta t = 1/32$, $h = \Delta t = 1/64$, respectively; we recall that $y_T^* = y_h^{\Delta t}(T)$ and that our methodology forces $y_h^{\Delta t}(T)$ to belong to H_{0h}^1, explaining the observed behaviour of the above function in the neighbourhood of Γ. On Figures 4 and 5 we have represented the functions $t \to \Vert u(t) \Vert_{L^2(\Gamma)}$ (...) and $t \to \Vert u^*(t) \Vert_{L^2(\Gamma)}$ (—) for $t \in (0,T)$ and $h = \Delta t = 1/32$, $h = \Delta t = 1/64$, respectively. Finally, on Figures 6 and 7 we have compared $f(x_1, 0.5)$ (...) and $f^*(x_1, 0.5)$ (—) for $x_1 \in (0,1)$ and $h = \Delta t = 1/32$, $h = \Delta t = 1/64$, respectively. Comparing these two figures shows that $h = \Delta t = 1/32$ provides a (slightly) better approximation than $h = \Delta t = 1/64$; this is in agreement with the results in Table 1.

The results obtained here compared favourably with those in Carthel *et al.* (1994) where the same test problem was solved by other methods, including a second-order accurate time discretization method close to that discussed in Section 1.8.6 for distributed control problems (see also Carthel (1994) for further results and comments).

2.6.2. Second test problem

If one uses the notation of Section 2.6.1 we have for this test problem $\Omega = (0,1) \times (0,1)$, $T = 1$, $s = 0$, $y_0 = 0$, $y_T(x_1, x_2) = \min(x_1, x_2, 1 - x_1, 1 - x_2)$ and $\nu = 1/2\pi^2$; unlike the test problem of Section 2.6.1, for which $\Gamma_0 = \Gamma$, we have here $\Gamma_0 \neq \Gamma$ since

$$\Gamma_0 = \{\{x_1, x_2\} \mid 0 < x_1 < 1,\ x_2 = 0\}.$$

The function y_T is *Lipschitz continuous*, but not smooth enough to have (see the discussion in Carthel *et al.* (1994, Section 2.3.3)) exact controllability.

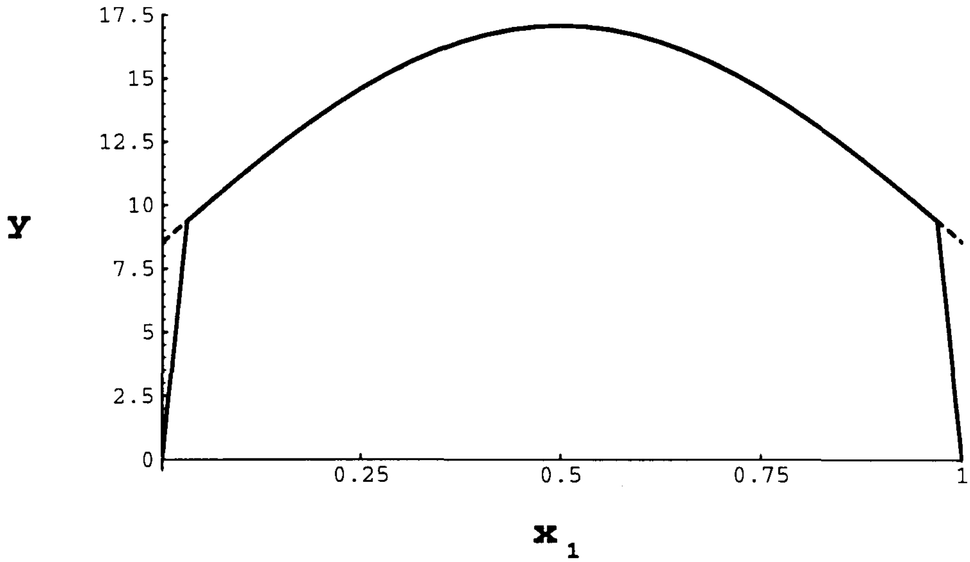


Fig. 2. Comparison between y_T (...) and y_T^* (—) ($h = \Delta t = 1/32$).

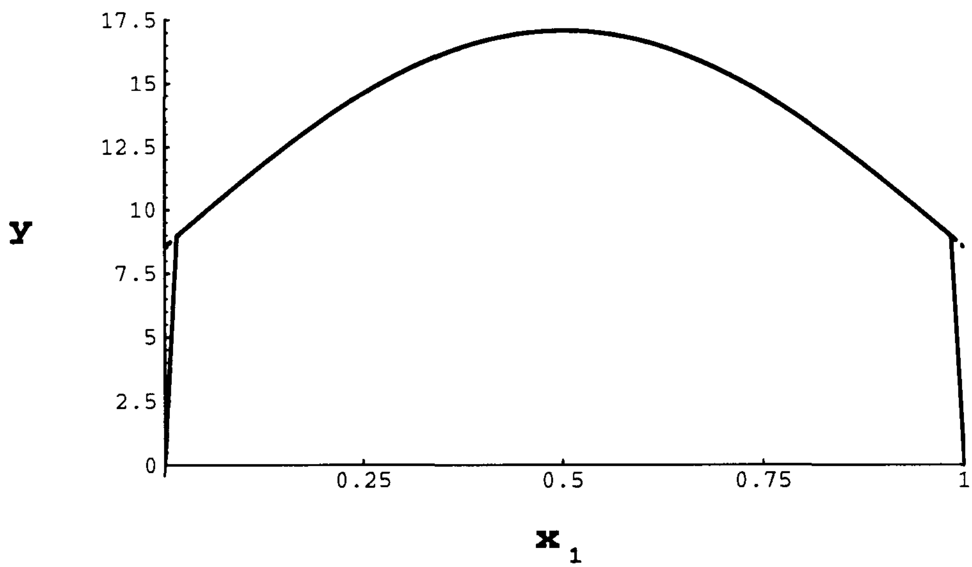


Fig. 3. Comparison between y_T (...) and y_T^* (—) ($h = \Delta t = 1/64$).

This implies that problem (2.21) has no solution if $k = +\infty$; on the other hand, problems (2.11), (2.12), (2.21), (2.25) are *well-posed* for any finite positive value of k or β. Focusing on the solution of problem (2.21) we have used the same space and time discretization methods as for the first test problem, with $h = \Delta t = 1/32$ and $h = \Delta t = 1/64$. We have taken $k = 10^5$

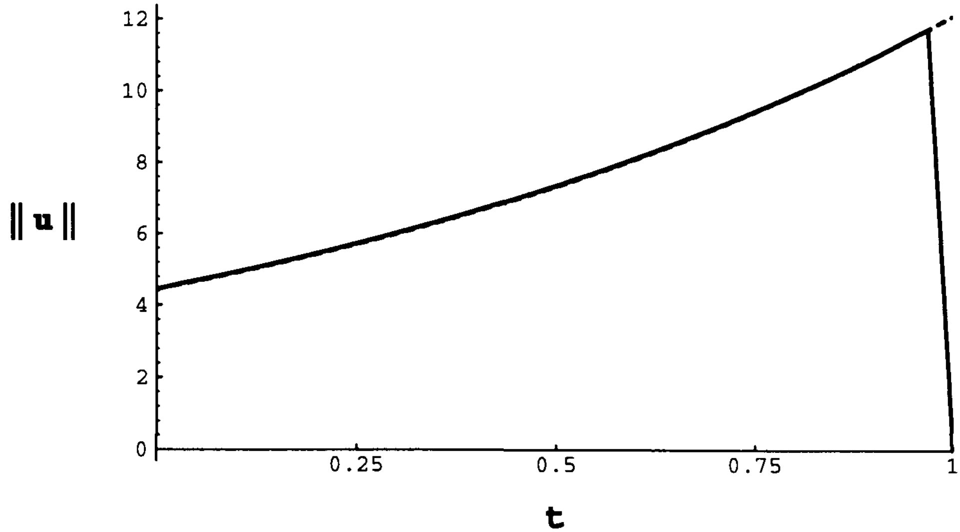


Fig. 4. Comparison between $\|u(t)\|_{L^2(\Gamma)}$ (...) and $\|u^*(t)\|_{L^2(\Gamma)}$ (—) ($h = \Delta t = 1/32$).

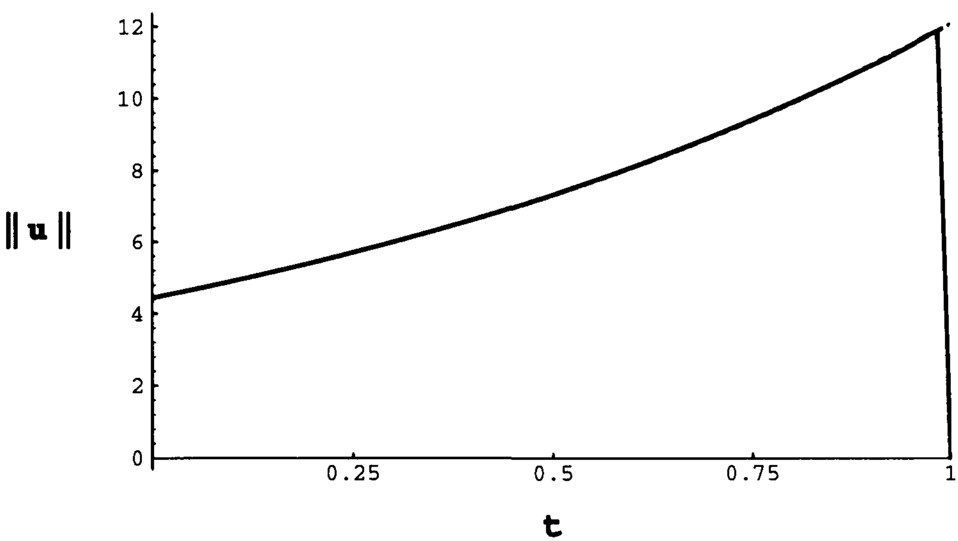


Fig. 5. Comparison between $\|u(t)\|_{L^2(\Gamma)}$ (...) and $\|u^*(t)\|_{L^2(\Gamma)}$ (—) ($h = \Delta t = 1/64$).

and 10^7 for the penalty parameter and used $\epsilon = 10^{-3}$ for the stopping criterium of the conjugate gradient algorithm (2.125)–(2.151) (which has been initialized with $f^0 = 0$).

The numerical results have been summarized in Table 2.

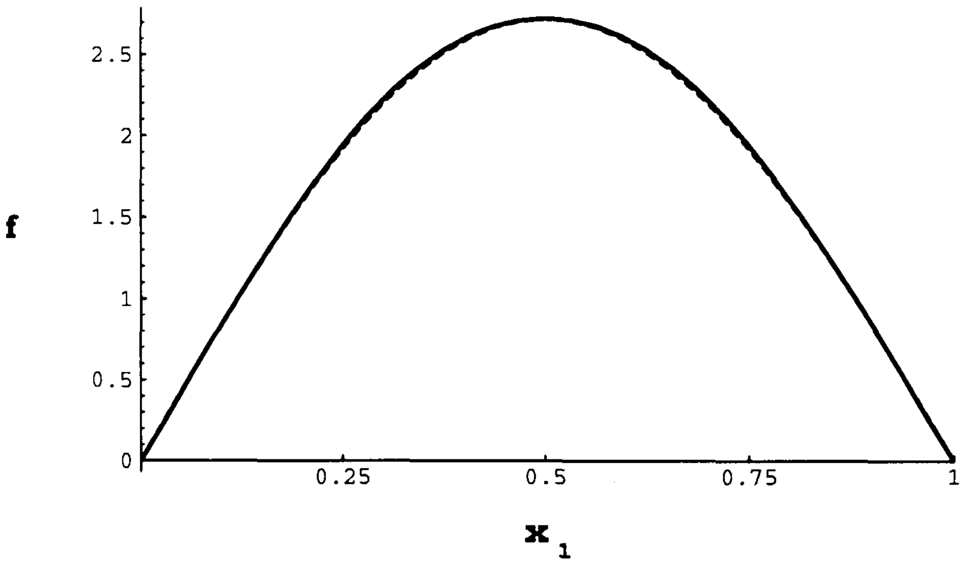


Fig. 6. Comparison between f (...) and f^* (—) ($h = \Delta t = 1/32$).

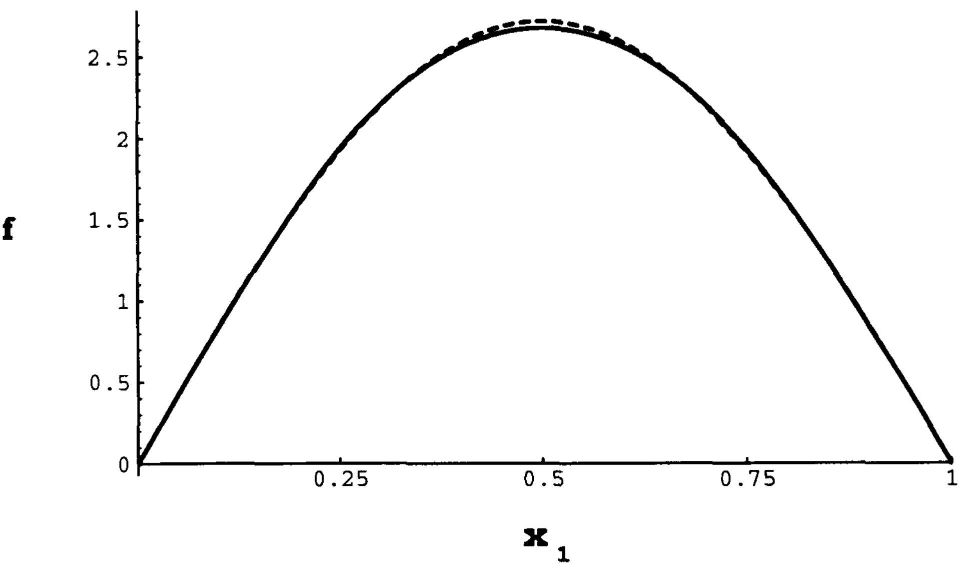


Fig. 7. Comparison between f (...) and f^* (—) ($h = \Delta t = 1/64$).

The above results suggest the following comments: first, we observe that $\|y_T - y_h^{\Delta t}(T)\|_{-1}$ varies like $k^{-1/4}$, approximately. Second, we observe that the number of iterations necessary for convergence, increases as h, Δt and k^{-1} decrease; there is no mystery here, since – from Section 1.8.2, relation

Table 2. *Summary of numerical results.*

$h = \Delta t$	1/32	1/64	1/32	1/64
k	10^5	10^5	10^7	10^7
Number of iterations	56	60	292	505
$\dfrac{\Vert y_T^* - y_T\Vert_{-1}}{\Vert y_T\Vert_{-1}}$	1.31×10^{-1}	1.28×10^{-1}	4.15×10^{-2}	3.93×10^{-2}
$\Vert u^*\Vert_{L^2(\Sigma_0)}$	8.18	8.12	25.59	24.78
$\Vert f^*\Vert_{H_0^1(\Omega)}$	600.4	584.2	18,960	17,950
$\Vert f^*\Vert_{L^2(\Omega)}$	75.95	73.63	1,632	1,525

(1.130) – the key factor controlling the speed of convergence is the *condition number* of the *bilinear form* in the left-hand side of equation (2.124). This condition number, denoted by $\nu_h^{\Delta t}(k)$, is defined by

$$\nu_h^{\Delta t}(k) = \max_{z \in H_{0h}^1 - \{0\}} R_h(z) \Big/ \min_{z \in H_{0h}^1 - \{0\}} R_h(z), \tag{2.161}$$

where, in (2.161), $R_h(z)$ is the *Rayleigh quotient* defined by

$$R_h(z) = \frac{k^{-1} \int_\Omega |\nabla z|^2 \, dx + \int_\Omega (\Lambda_h^{\Delta t} z) z \, dx}{\int_\Omega |\nabla z|^2 \, dx}; \tag{2.162}$$

it can be shown that

$$\lim_{\substack{h \to 0 \\ \Delta t \to 0}} \nu_h^{\Delta t}(k) = k \|\Lambda\|_{\mathcal{L}(H_0^1(\Omega); H^{-1}(\Omega))}, \tag{2.163}$$

implying that for small values of h, Δt and k^{-1}, problem (2.124) is *badly conditioned.* Indeed, we can expect from (2.163) and from Section 1.8.2, relation (1.130), that for h and Δt sufficiently small the number of iterations necessary to obtain convergence will vary like $k^{1/2}$, approximately; this prediction is confirmed by the results in Table 2 (and will be further confirmed by the results in Section 2.6.3, Table 3, concerning our third test problem). Third, and finally, we observe that $\|u^*\|_{L^2(\Sigma_0)}$ (respectively $\|f^*\|_{H_0^1(\Omega)}$) varies like $k^{1/4}$ (respectively $k^{3/4}$); it can be shown that the behaviour of $\|f^*\|_{H_0^1(\Omega)}$ follows from that of $\|y_T - y_T^*\|_{-1}$ since we have (see, e.g., Carthel *et al.* (1994, Remark 4.3))

$$k^{-1} = \frac{\|y_T - y(T)\|_{-1}}{\|f\|_{H_0^1(\Omega)}}. \tag{2.164}$$

where y is the state function obtained from the optimal control u via (2.1)–(2.3).

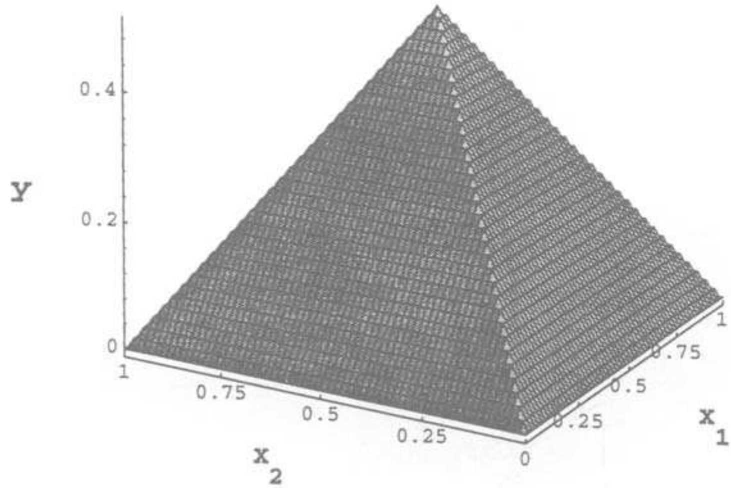


Fig. 8. Graph of the target function y_T ($y_T(x_1, x_2) = \min(x_1, x_2, 1 - x_1, 1 - x_2), 0 \leq x_1, x_2 \leq 1$).

On the following figures, we have represented or shown the following information and results.

A view of the target function y_T on Figure 8. On Figures 9(a) to 12(a) (respectively 9(b) to 12(b)) the graph of the function $y_T^* (= y_h^{\Delta t}(T))$ (respectively a comparison between y_T (...) and the actually reached state function y_T^* (—)) for various values of h, Δt and k (we have shown the graphs of the functions $x_2 \to y_T(0.5, x_2)$ and $x_2 \to y_T^*(0.5, x_2)$ for $x_2 \in (0, 1)$). The graphs of the computed solution $f_h^{\Delta t} (= f^*)$ and of the function $x_2 \to f_h^{\Delta t}(0.5, x_2)$ on Figures 13 to 16. On Figures 17 to 20 the graphs of the functions $t \to \|u^*(t)\|_{L^2(\Gamma_0)}$ and $\{x_1, t\} \to u^*(x_1, t)$. Finally, we have visualized on Figures 21 to 24 (using a log-scale) the convergence to zero of the conjugate gradient residual $\|g_m\|_{H_0^1(\Omega)}$; the observed behaviour (highly oscillatory, particularly for $k = 10^7$) is typical of a *badly conditioned* problem.

2.6.3. Third test problem

For this test problem Ω, T, Γ_0, y_0, s, A, ν are as in Section 2.6.2, namely $\Omega = (0.1) \times (0.1)$, $T = 1$, $y_0 = 0$, $s = 0$, $A = -\nu\Delta$ with $\nu = 1/2\pi^2$; the only difference is that this time y_T is the *discontinuous* function defined by

$$y_T(x_1, x_2) = \begin{cases} 1 & \text{if } 1/4 < x_1, x_2 < 3/4, \\ 0 & \text{otherwise.} \end{cases} \tag{2.165}$$

We have applied to this problem the solution methods considered in Section 2.6.2; their behaviour here is essentially the same as that for the test problem of Section 2.6.2 (where y_T was Lipschitz continuous). We have shown in the

Table 3. *Summary of numerical results.*

$h = \Delta t$	1/32	1/64	1/32	1/64
k	10^5	10^5	10^7	10^7
Number of iterations	55	56	361	569
$\frac{\lVert y_T^* - y_T \rVert_{-1}}{\lVert y_T \rVert_{-1}}$	1.64×10^{-1}	1.57×10^{-1}	1.05×10^{-1}	9.88×10^{-2}
$\lVert u^* \rVert_{L^2(\Sigma_0)}$	14.68	15.07	56.80	58.53
$\lVert f^* \rVert_{H_0^1(\Omega)}$	1,407	1,410	90,010	88,510
$\lVert f^* \rVert_{L^2(\Omega)}$	120.7	122.5	5,608	5,566

following Table 3 the results of our numerical experiments (the notation is as in Section 2.6.2).

Comparing to Table 2 we observe that the convergence properties of the conjugate gradient algorithm are essentially the same, despite the fact that y_T is much less smooth here; on the other hand we observe that $\lVert y_T - y_h^{\Delta t}(T) \rVert_{-1}$ varies like $k^{-1/3}$, approximately, implying in turn (from (2.164)) that $\lVert f^* \rVert_{H_0^1(\Omega)}$ varies like $k^{7/8}$, approximately. The dependence of $\lVert u^* \rVert_{L^2(\Sigma_0)}$ is less clear (to us at least); it looks 'faster', however, than $k^{1/4}$.

On Figure 25 we have visualized the graph of the target function y_T, then on Figures 26 and 27 we have compared the function $x_2 \to y_T(0.5, x_2)$ to $x_2 \to y_T^*(0.5, x_2)$ (—) for various values of k, h and Δt; on Figures 28 and 29 we have shown the graphs of the corresponding function y_T^*. Finally, for the above values of k, h and Δt, we have shown, on Figures 30 to 35, further information concerning $u_h^{\Delta t}$, $f_h^{\Delta t}$ and the convergence of the conjugate gradient algorithm (2.125)–(2.151).

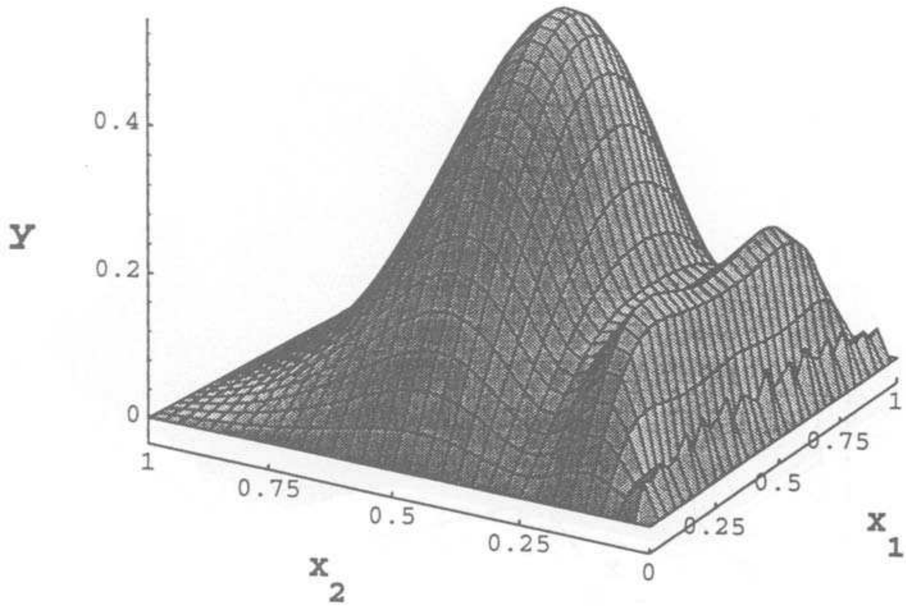
2.7. Neumann control (I): Formulation of the control problems

We consider again the state equation (2.1) in Q and the initial condition (2.3). We suppose this time that the *boundary control* is of the Neumann's type. To be more precise, the state function y is defined now by

$$\frac{\partial y}{\partial t} + Ay = 0 \text{ in } Q, \quad y(0) = 0, \; \frac{\partial y}{\partial n_A} = v \text{ on } \Sigma_0, \; \frac{\partial y}{\partial n_A} = 0 \text{ on } \Sigma \backslash \Sigma_0. \quad (2.166)$$

In (2.166), $\partial/\partial n_A$ denotes the *conormal derivative* operator; if operator A is defined by

$$A\varphi = -\sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial \varphi}{\partial x_j}, \quad (2.167)$$



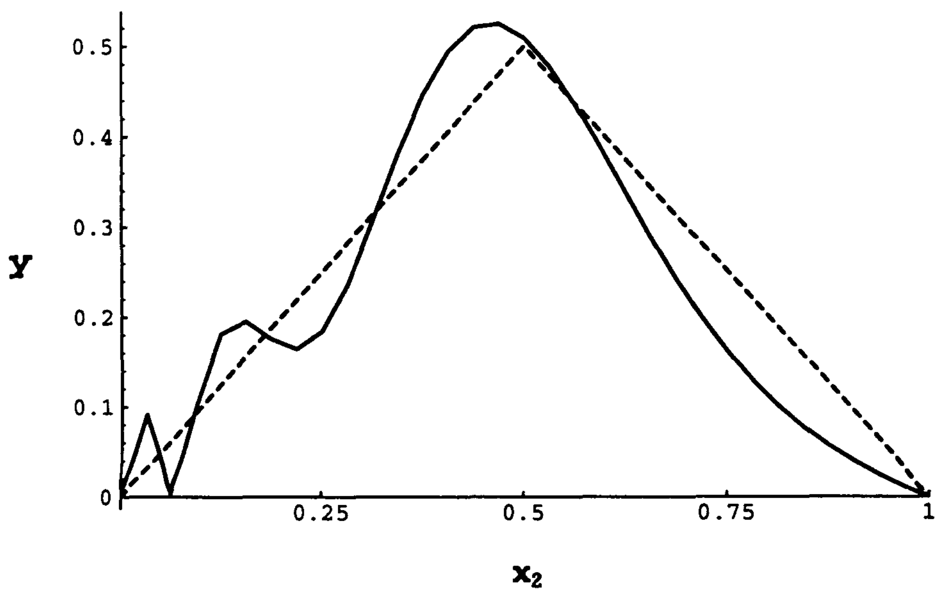


Fig. 9. (a) Graph of the function $y_T^*(k = 10^5,\ h = \Delta t = 1/32)$. (b) Comparison between y_T (...) and y_T^* (—) $(k = 10^5,\ h = \Delta t = 1/32)$.

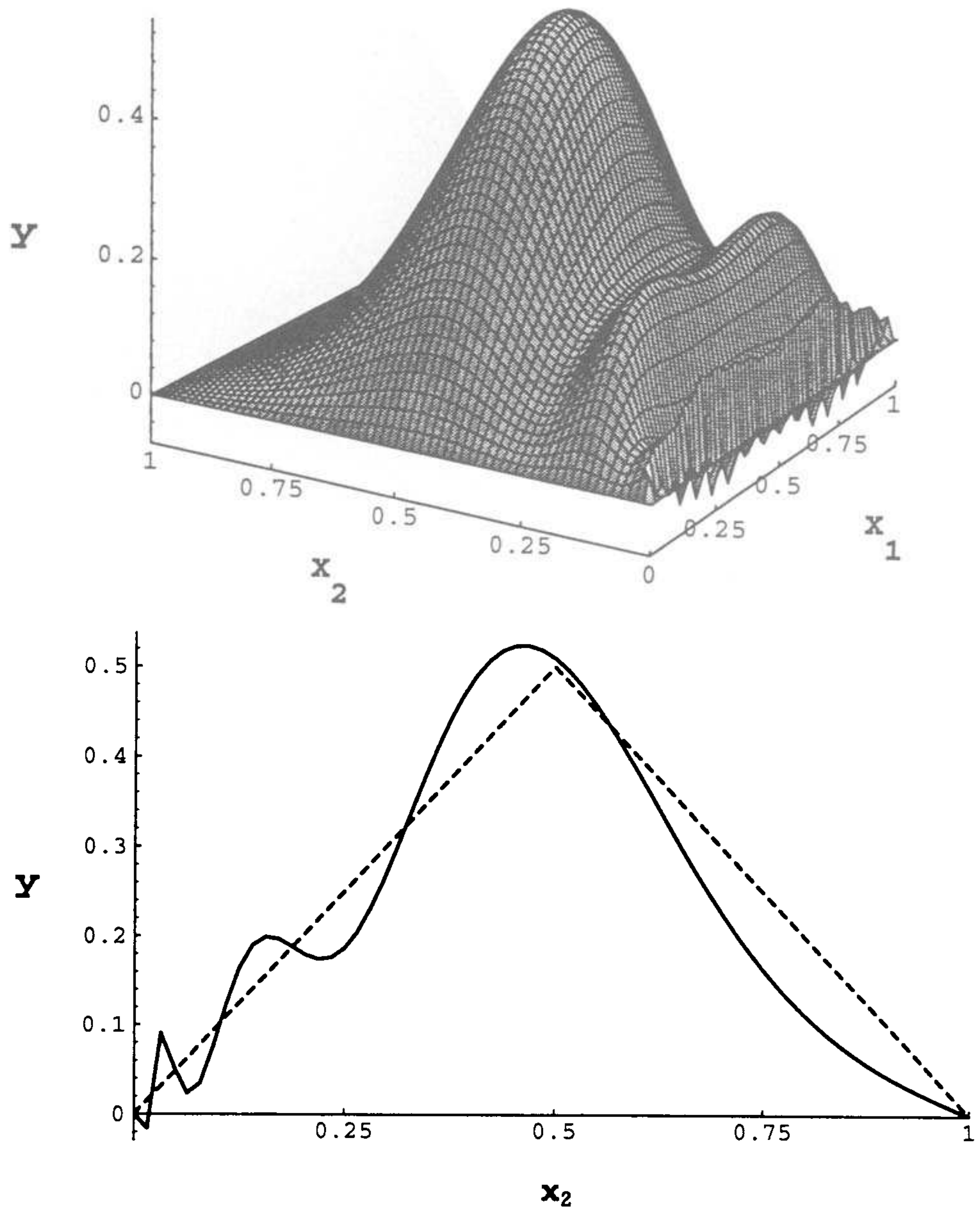


Fig. 10. (a) Graph of the function $y_T^*(k = 10^5,\ h = \Delta t = 1/64)$. (b) Comparison between y_T (...) and y_T^* (—) $(k = 10^5,\ h = \Delta t = 1/64)$.

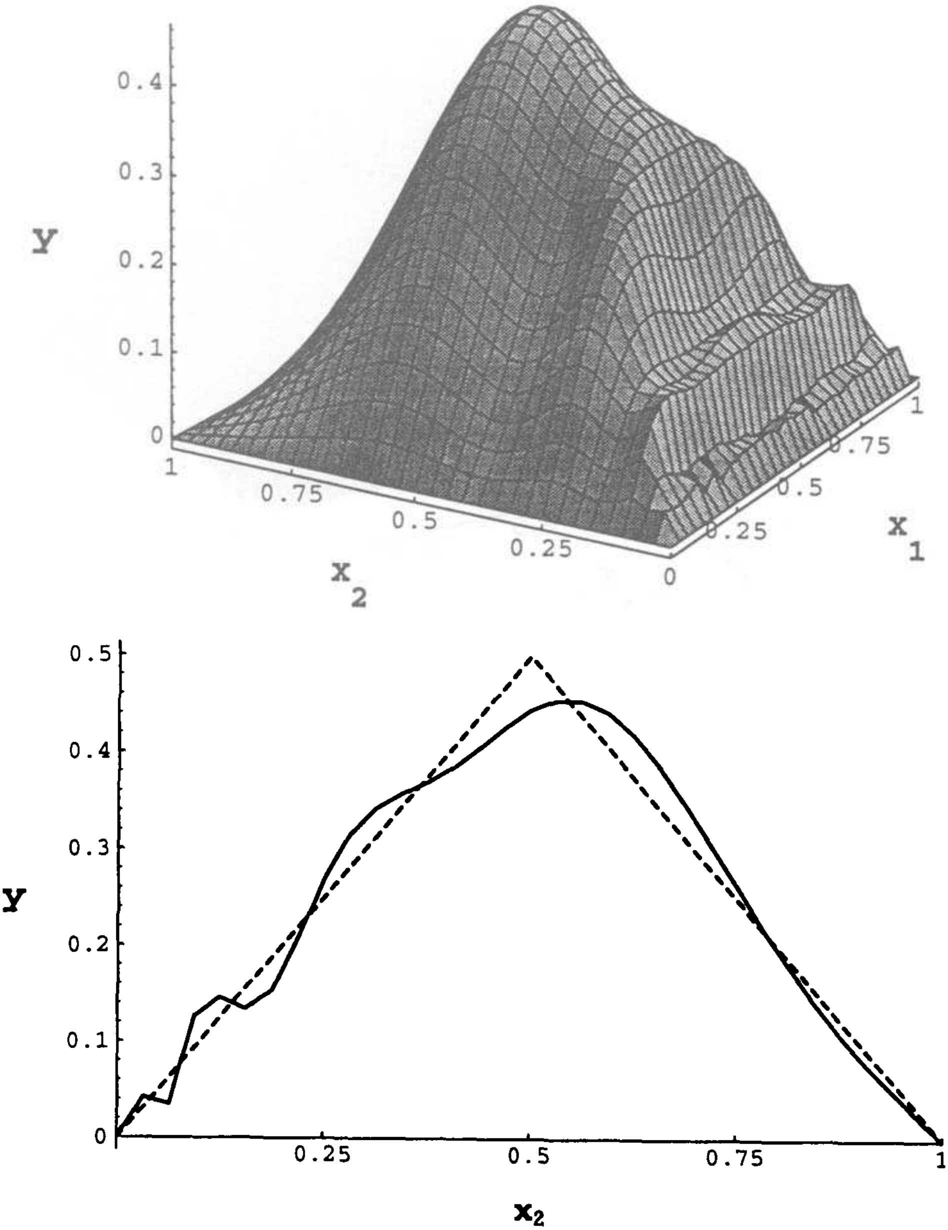


Fig. 11. (a) Graph of the function $y_T^*(k = 10^7,\ h = \Delta t = 1/32)$. (b) Comparison between y_T (...) and y_T^* (—) $(k = 10^7,\ h = \Delta t = 1/32)$.

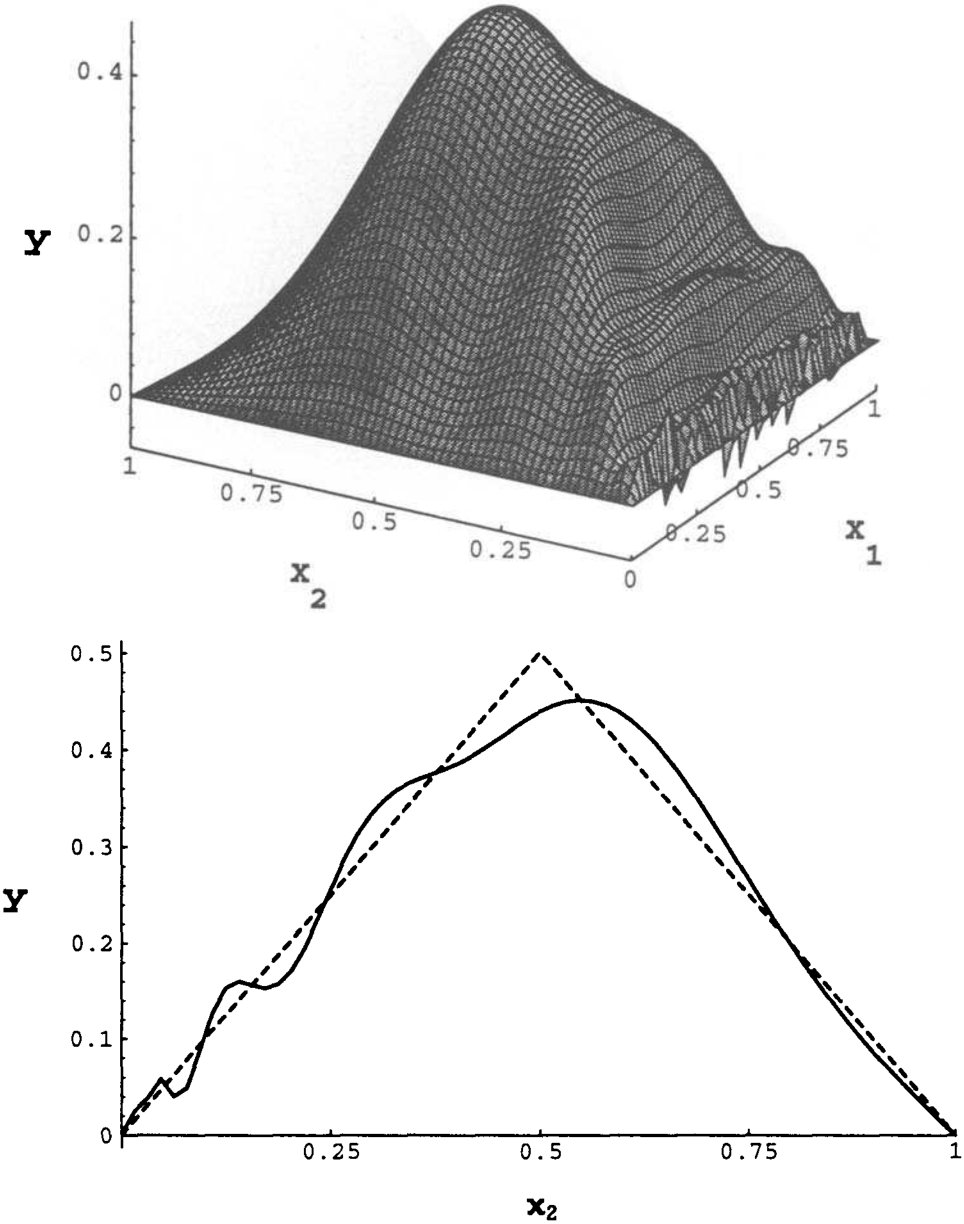
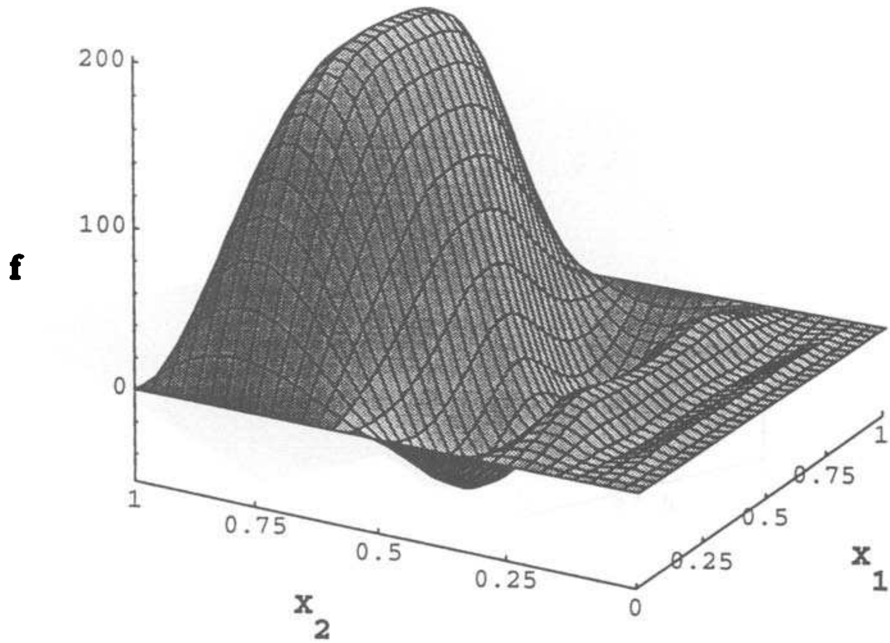


Fig. 12. (a) Graph of the function $y_T^*(k = 10^7,\ h = \Delta t = 1/64)$. (b) Comparison between y_T (...) and y_T^* (—) $(k = 10^7,\ h = \Delta t = 1/64)$.



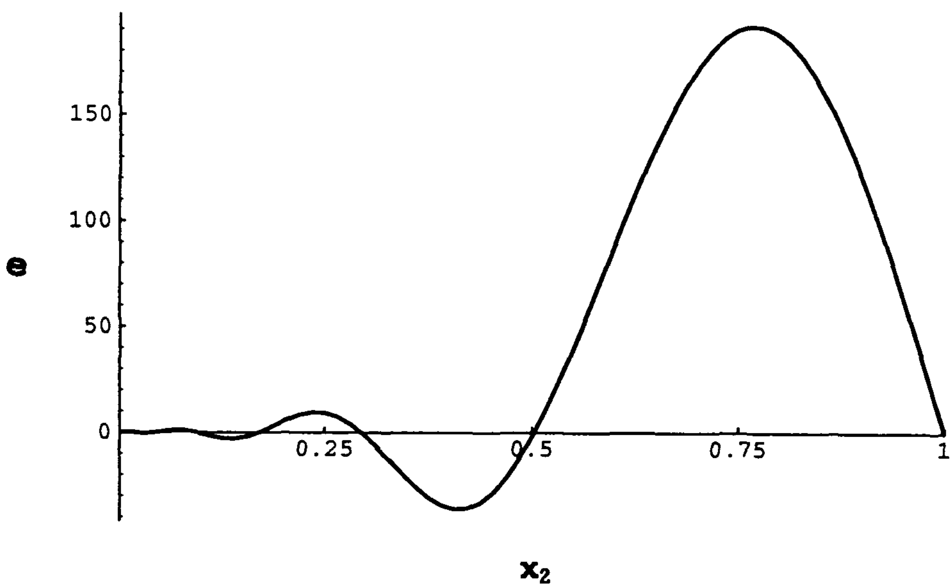


Fig. 13. (a) Graph of the function $f_h^{\Delta t}(k = 10^5,\ h = \Delta t = 1/32)$. (b) Graph of the function $x_2 \to f_h^{\Delta t}(0.5, x_2)(k = 10^5,\ h = \Delta t = 1/32)$.

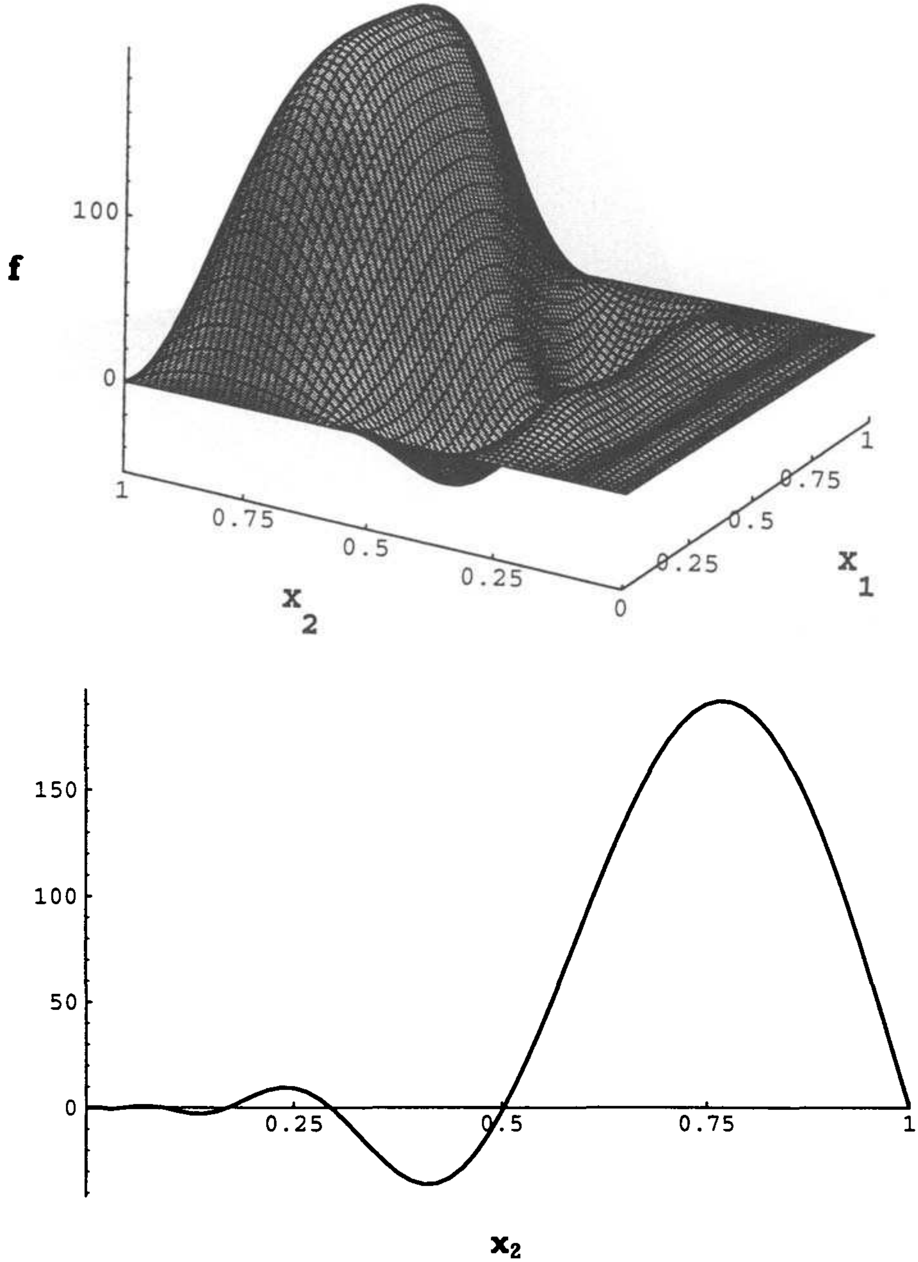


Fig. 14. (a) Graph of the function $f_h^{\Delta t}(k = 10^5,\ h = \Delta t = 1/64)$. (b) Graph of the function $x_2 \to f_h^{\Delta t}(0.5, x_2)(k = 10^5,\ h = \Delta t = 1/64)$.

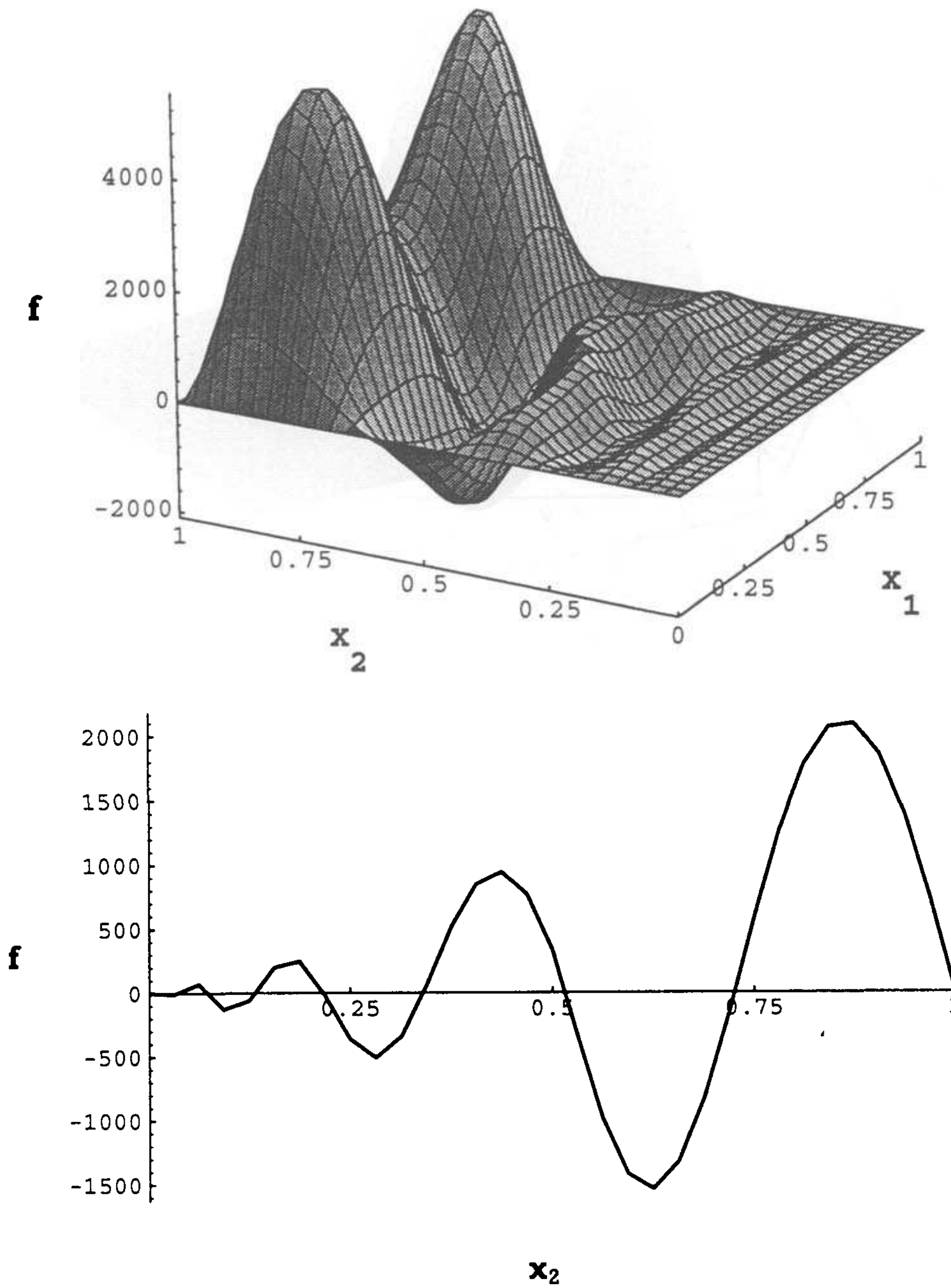


Fig. 15. (a) Graph of the function $f_h^{\Delta t}(k = 10^7,\ h = \Delta t = 1/32)$. (b) Graph of the function $x_2 \to f_h^{\Delta t}(0.5, x_2)(k = 10^7,\ h = \Delta t = 1/32)$.

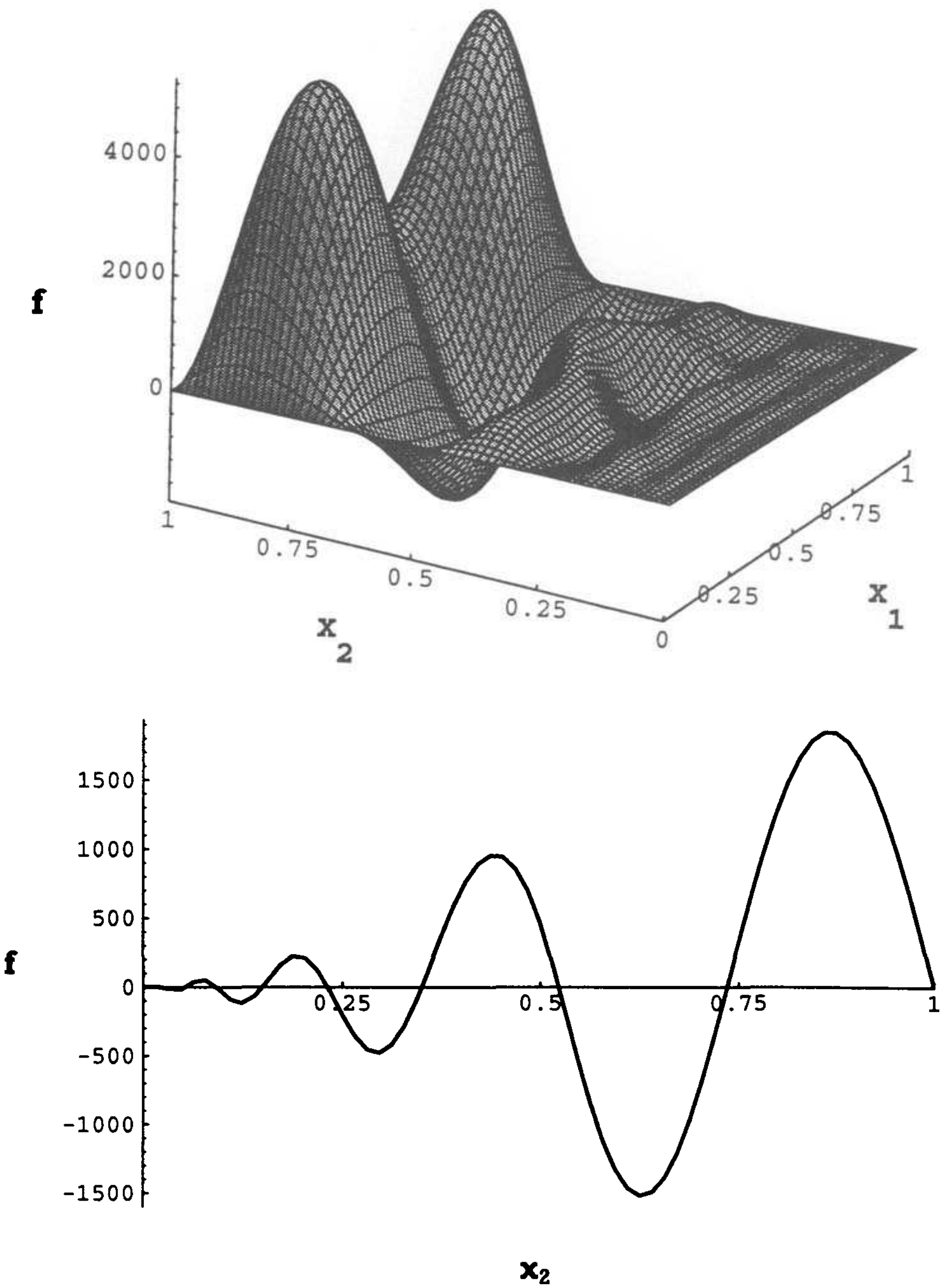


Fig. 16. (a) Graph of the function $f_h^{\Delta t}(k = 10^7,\ h = \Delta t = 1/64)$. (b) Graph of the function $x_2 \to f_h^{\Delta t}(0.5, x_2)(k = 10^7,\ h = \Delta t = 1/64)$.

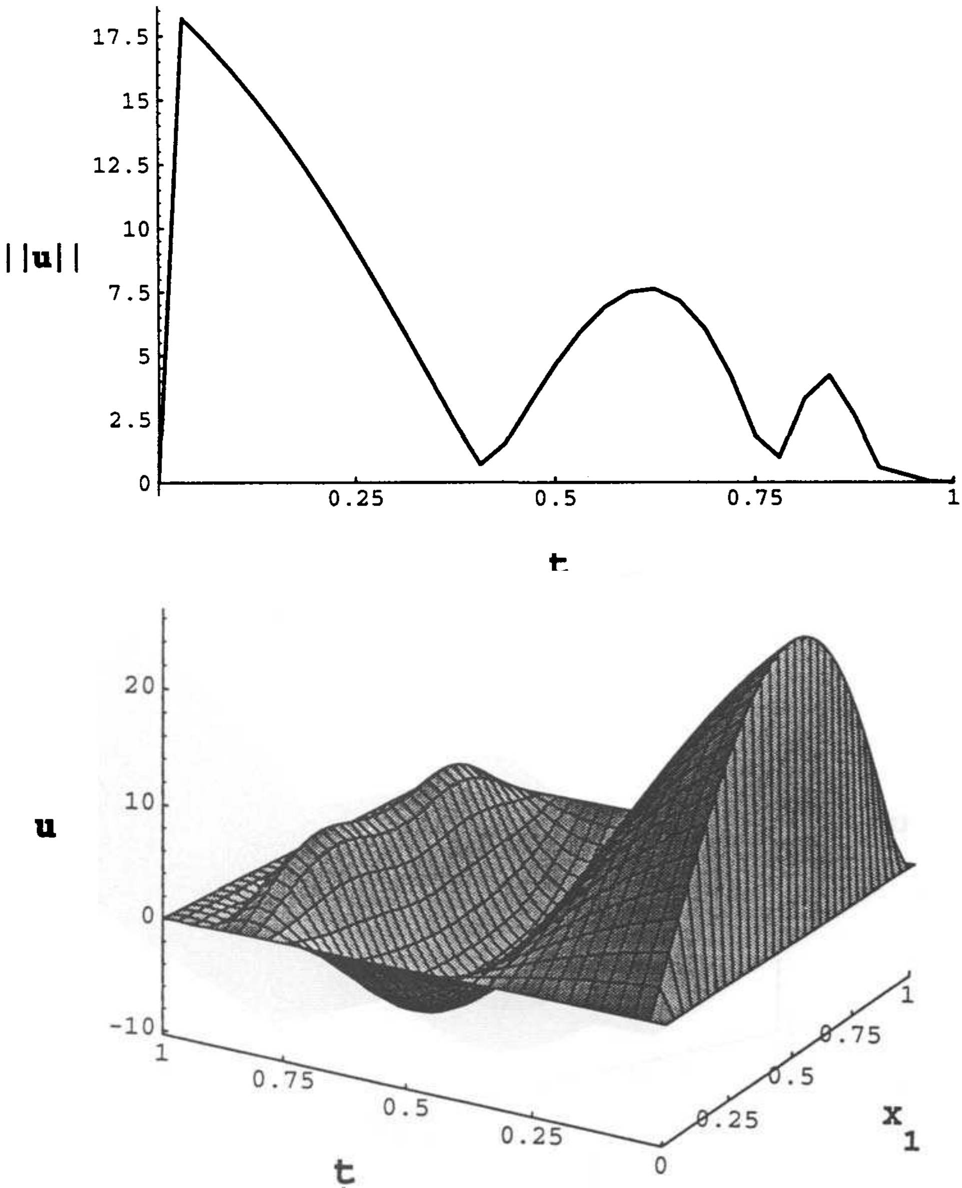
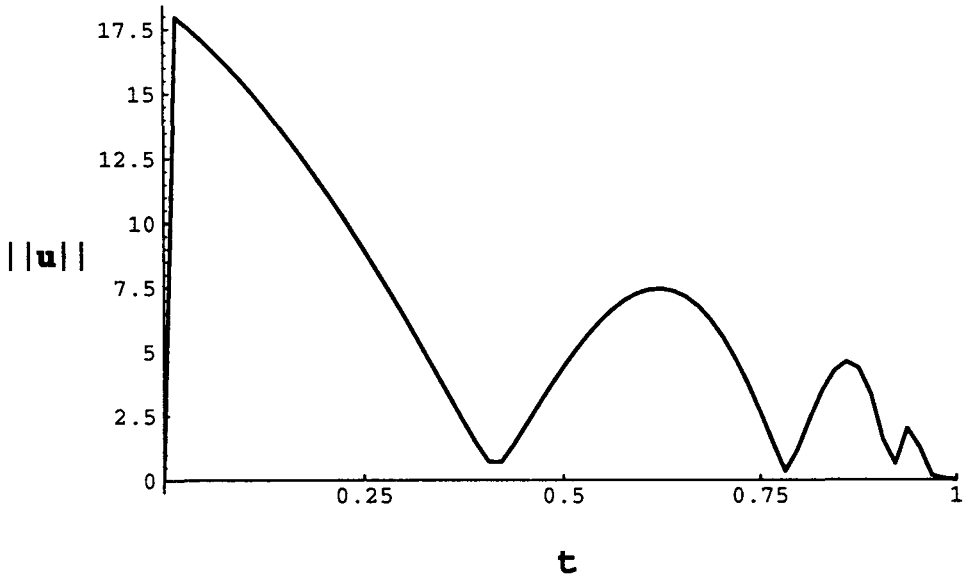


Fig. 17. (a) Graph of $t \to \|u^*(t)\|_{L^2(\Gamma_0)}(k = 10^5,\ h = \Delta t = 1/32)$. (b) Graph of the computed boundary control $(k = 10^5,\ h = \Delta t = 1/32)$.



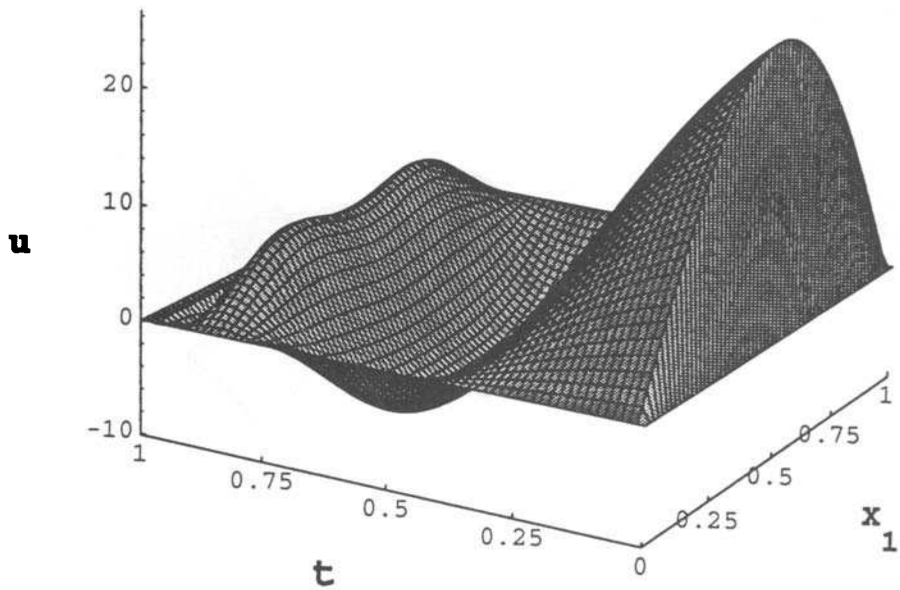


Fig. 18. (a) Graph of $t \to \|u^*(t)\|_{L^2(\Gamma_0)}(k = 10^5,\ h = \Delta t = 1/64)$. (b) Graph of the computed boundary control ($k = 10^5,\ h = \Delta t = 1/64$).

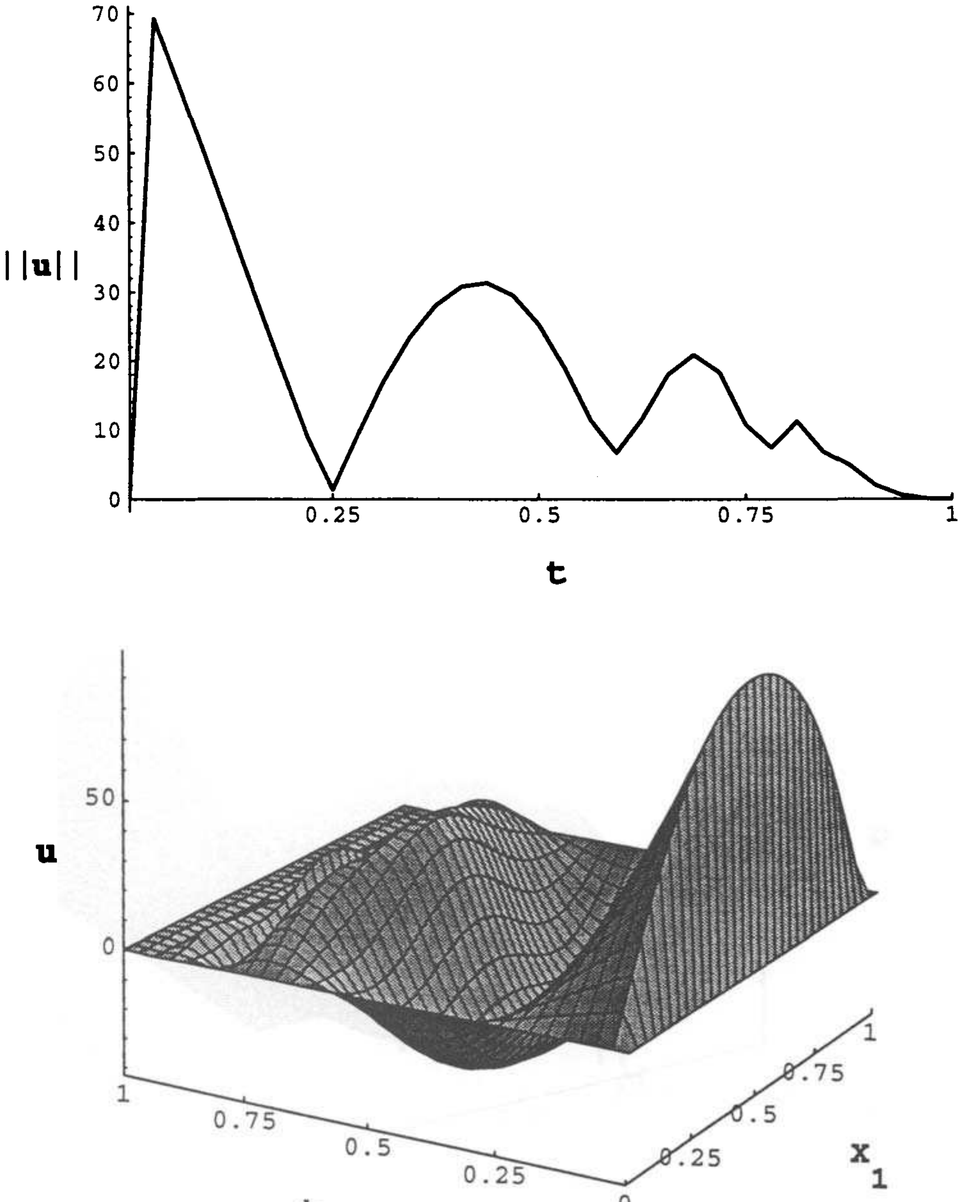


Fig. 19. (a) Graph of $t \to \|u^*(t)\|_{L^2(\Gamma_0)}(k = 10^7,\ h = \Delta t = 1/32)$. (b) Graph of the computed boundary control $(k = 10^7,\ h = \Delta t = 1/32)$.

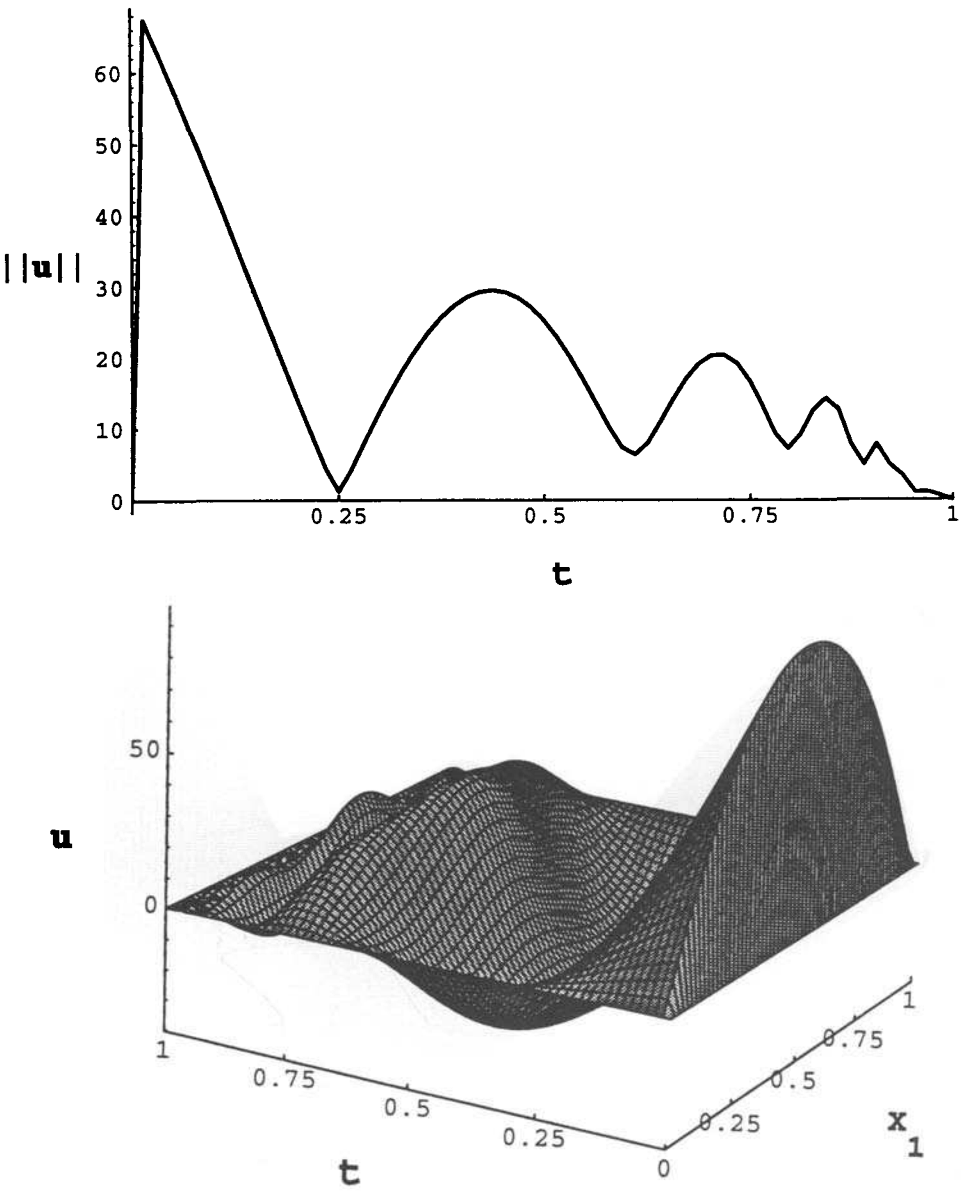


Fig. 20. (a) Graph of $t \to \|u^*(t)\|_{L^2(\Gamma_0)}(k = 10^7,\ h = \Delta t = 1/64)$. (b) Graph of the computed boundary control ($k = 10^7,\ h = \Delta t = 1/64$).

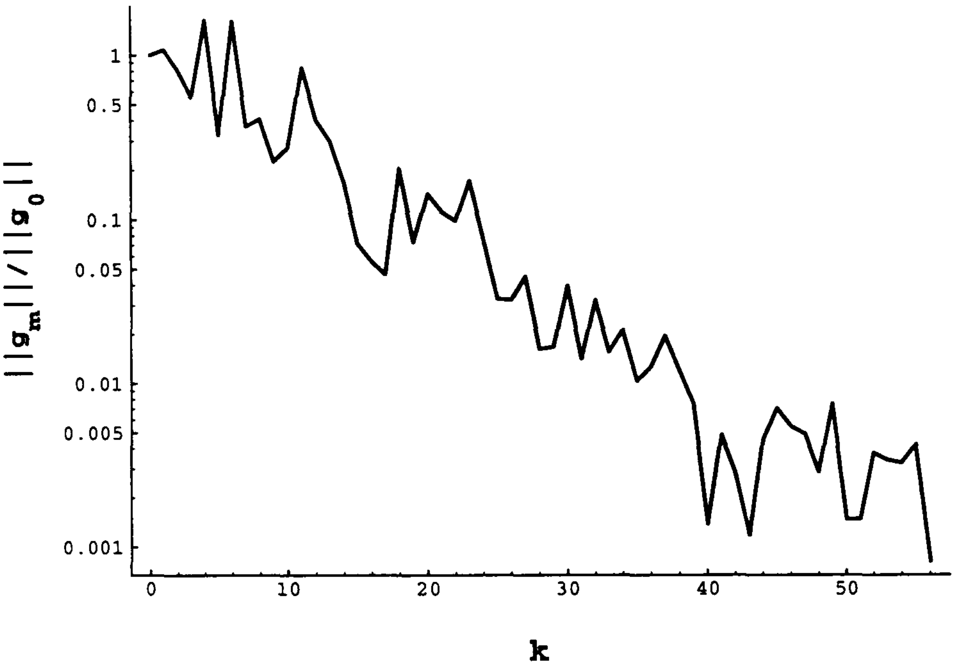


Fig. 21. Variation of $\|g_m\|_{H_0^1(\Omega)}/\|g_0\|_{H_0^1(\Omega)}(k = 10^5,\ h = \Delta t = 1/32)$.

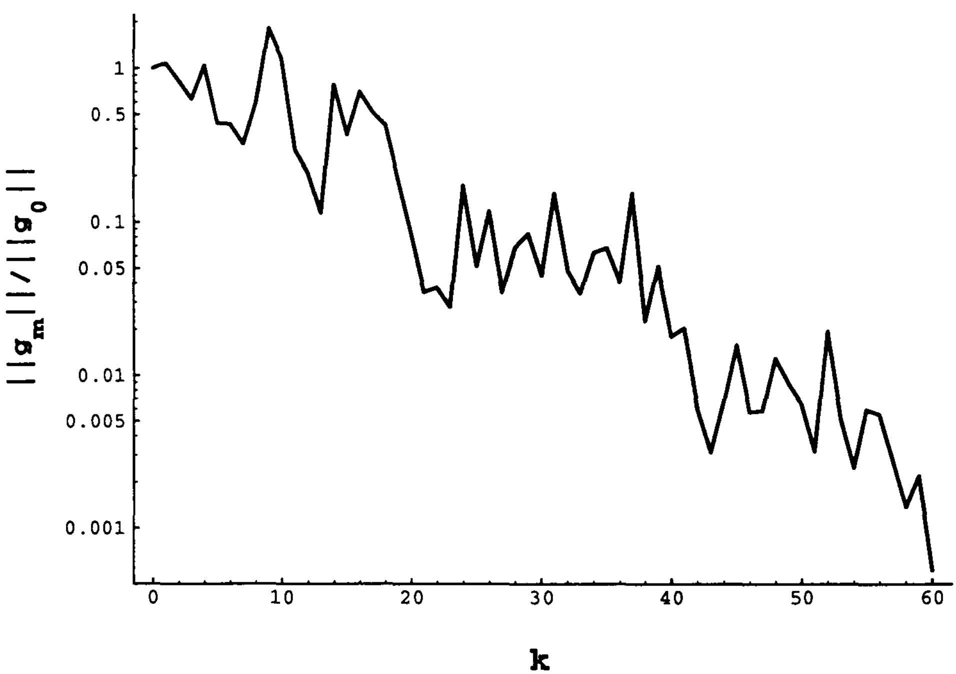


Fig. 22. Variation of $\|g_m\|_{H_0^1(\Omega)}/\|g_0\|_{H_0^1(\Omega)}(k = 10^5,\ h = \Delta t = 1/64)$.

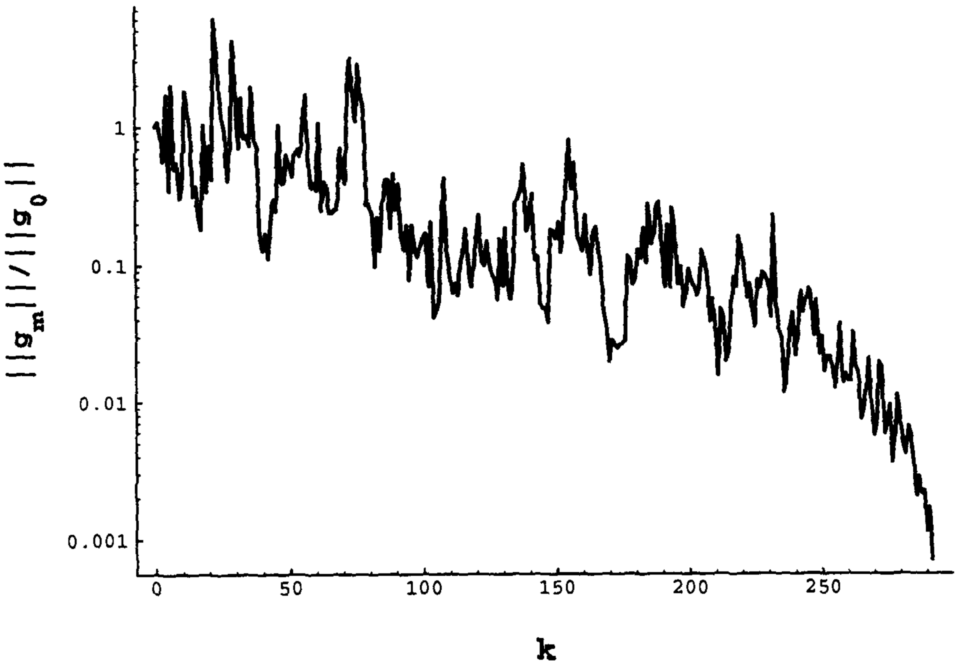


Fig. 23. Variation of $\|g_m\|_{H_0^1(\Omega)}/\|g_0\|_{H_0^1(\Omega)} (k = 10^7,\ h = \Delta t = 1/32)$.

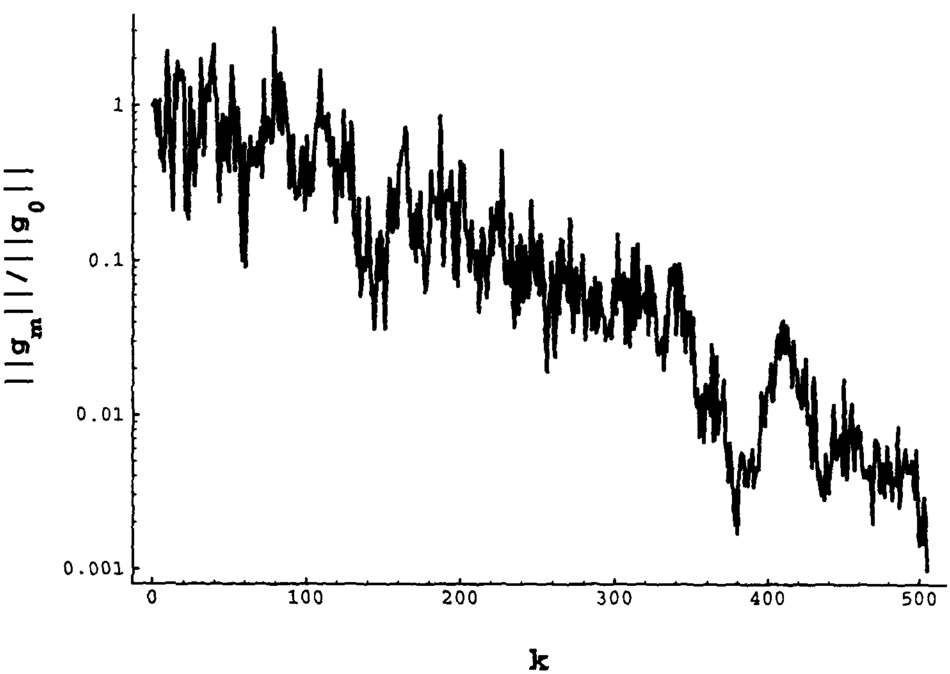


Fig. 24. Variation of $\|g_m\|_{H_0^1(\Omega)}/\|g_0\|_{H_0^1(\Omega)} (k = 10^7,\ h = \Delta t = 1/64)$.

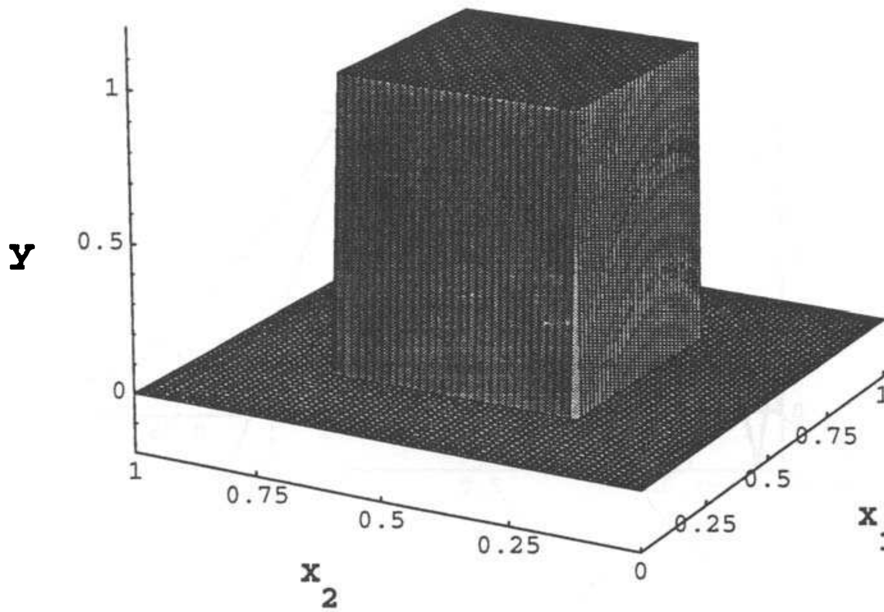


Fig. 25. Graph of the target function y_T (y_T is the characteristic function of the square $(1/4,3/4)^2$).

then $\partial/\partial n_A$ is defined by

$$\frac{\partial \varphi}{\partial n_A} = \sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}\frac{\partial \varphi}{\partial x_j}n_i, \tag{2.168}$$

where $\mathbf{n} = \{n_i\}_{i=1}^d$ is the *unit vector* of the *outward normal* at Γ.

We assume that

$$v \in L^2(\Sigma_0). \tag{2.169}$$

There are slight (and subtle) technical differences between Neumann and Dirichlet boundary controls. Indeed, suppose that operator A is defined by (2.167) with the following additional properties

$$a_{ij} \in L^\infty(\Omega),\ \forall 1 \leq i,j \leq d, \tag{2.170}$$

$$\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \alpha|\boldsymbol{\xi}|^2,\ \forall \boldsymbol{\xi} \in \mathbb{R}^d, \quad \text{a.e. in } \Omega, \text{ with } \alpha > 0 \tag{2.171}$$

(in (2.171), $|\boldsymbol{\xi}|^2 = \sum_{i=1}^d |\xi_i|^2$, $\forall \boldsymbol{\xi} = \{\xi_i\}_{i=1}^d \in \mathbb{R}^d$); then problem (2.166) can

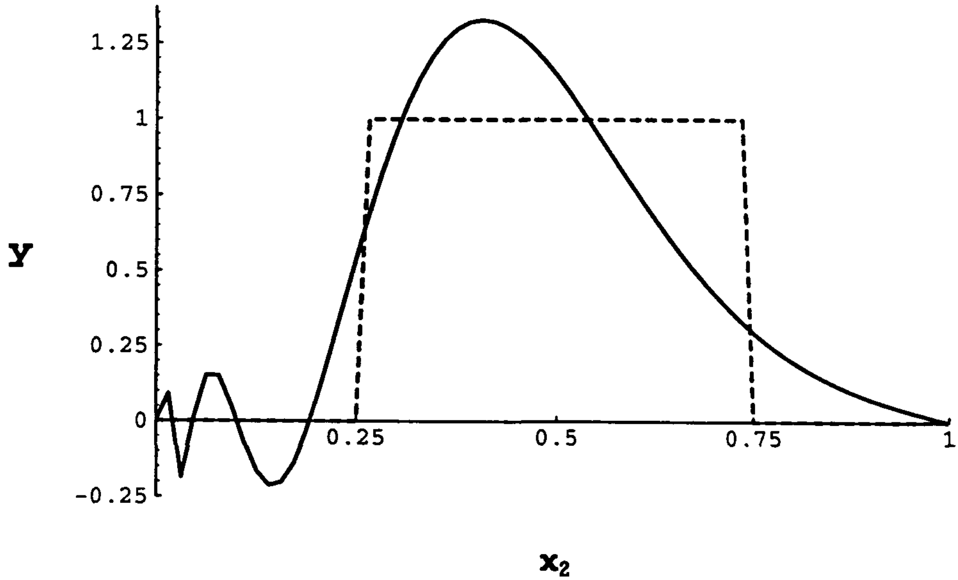


Fig. 26. Comparison between y_T (...) and y_T^* (—) ($k = 10^5$, $h = \Delta t = 1/64$).

be expressed in *variational* form as follows

$$\begin{cases} \left(\dfrac{\partial y}{\partial t}, \hat{y}\right) + a(y, \hat{y}) = \displaystyle\int_{\Gamma_0} v\hat{y}\, \mathrm{d}\Gamma, \ \forall \hat{y} \in H^1(\Omega), \\ y(t) \in H^1(\Omega) \ \text{ a.e. on } (0, T), \ y(0) = 0, \end{cases} \tag{2.172}$$

where

$$a(y, \hat{y}) = \sum_{i=1}^{d} \sum_{j=1}^{d} \int_\Omega a_{ij} \frac{\partial y}{\partial x_j} \frac{\partial \hat{y}}{\partial x_i} \, \mathrm{d}x \tag{2.173}$$

(actually, all this applies to the case where the coefficients a_{ij} depend on x *and* t and verify $a_{ij}(x, t) \in L^\infty(Q)$ and

$$\sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij}(x, t)\xi_i \xi_j \geq \alpha |\boldsymbol{\xi}|^2, \ \forall \boldsymbol{\xi} \in \mathbb{R}^d, \ \text{a.e. in } Q$$

with $\alpha > 0$). Therefore, without any further hypothesis on the coefficients a_{ij}, problem (2.166) admits a unique solution $y(v) = y(x, t; v)$ such that

$$y(v) \in L^2(0, T; H^1(\Omega)) \cap C^0([0, T]; L^2(\Omega)). \tag{2.174}$$

To obtain the *approximate controllability* property we shall assume further *regularity* properties for the a_{ij}'s, more specifically we shall assume that

$$a_{ij} \in C^1(\bar{\Omega}), \ \forall 1 \leq i, j \leq d. \tag{2.175}$$

We have then the following

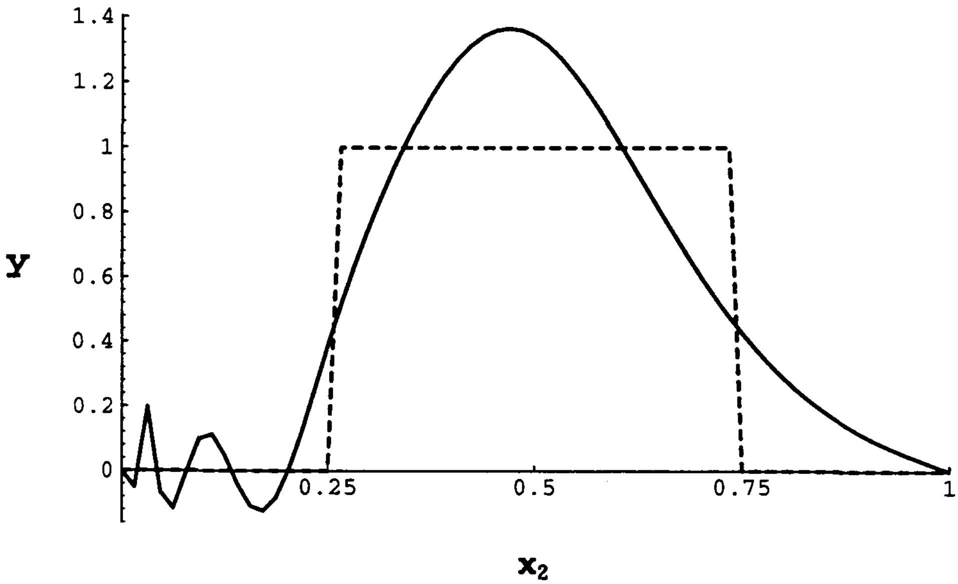


Fig. 27. Comparison between y_T (...) and y_T^* (—) ($k = 10^7$, $h = \Delta t = 1/64$).

Proposition 2.3 *Suppose that coefficients a_{ij} verify* (2.171) *and* (2.175). *Then $y(T; v)$ spans a dense subset of $L^2(\Omega)$ when v spans $L^2(\Sigma_0)$.*

Proof. The proof is similar to the proof of Proposition 2.1 (see Section 2.1). Let us assume therefore that $f \in L^2(\Omega)$ satisfies

$$\int_\Omega y(T; v) f \, dx = 0, \ \forall v \in L^2(\Sigma_0). \tag{2.176}$$

We introduce ψ as the solution of

$$-\frac{\partial \psi}{\partial t} + A^* \psi = 0 \text{ in } Q, \quad \psi(T) = f, \ \frac{\partial \psi}{\partial n_{A^*}} = 0 \text{ on } \Sigma; \tag{2.177}$$

then (2.176) is equivalent to

$$\psi = 0 \text{ on } \Sigma_0. \tag{2.178}$$

Thanks to the regularity hypothesis (2.175) we can use the *Mizohata's uniqueness theorem* (Mizohata, 1958) (see also Saut and Schoerer (1987)): it follows then from (2.177) and (2.178) that $\psi = 0$, hence $f = 0$ and the proof is completed.

Remark 2.8 The applicability of the Mizohata uniqueness theorem under the only assumption that $a_{ij} \in L^\infty(Q)$ does not seem to have been completely settled, yet.

We can state two basic *controllability problems* both closely related to problems (2.11) and (2.12) in Section 2.1.

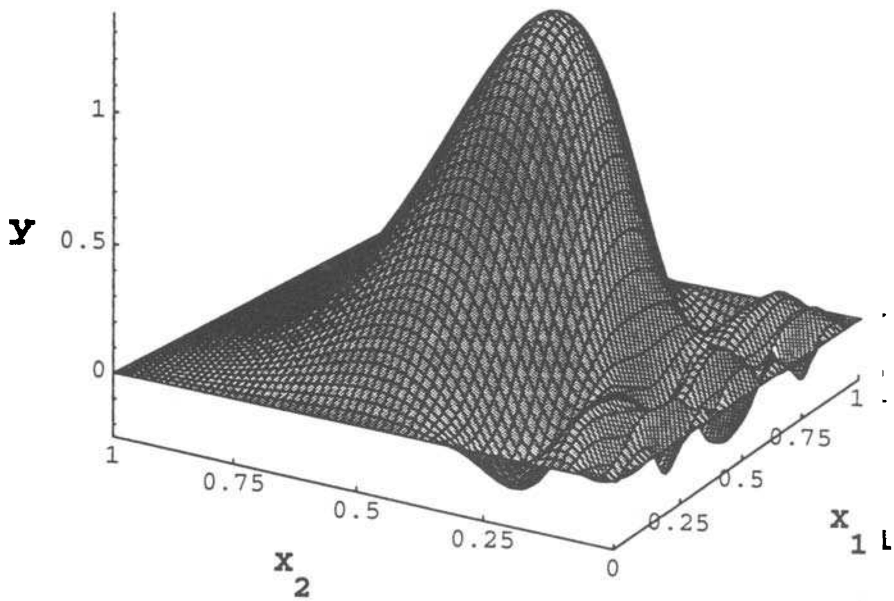


Fig. 28. Graph of the function $y_T^*(k = 10^5,\ h = \Delta t = 1/64)$.

The *first* Neumann control problem that we consider is defined by

$$\inf_v \frac{1}{2} \int_{\Sigma_0} v^2 \,\mathrm{d}\Sigma, \tag{2.179}$$

where v is subjected to

$$y(T; v) \in y_T + \beta B; \tag{2.180}$$

in (2.180), $y(t; v)$ is the solution of problem (2.166), the target function y_T belongs to $L^2(\Omega)$, B denotes the closed unit ball of $L^2(\Omega)$ and β is a positive number, arbitrarily small.

The *second* Neumann control problem to be considered is defined by

$$\inf_v \left[\frac{1}{2} \int_{\Sigma_0} v^2 \,\mathrm{d}\Sigma + \frac{1}{2} k \|y(T; v) - y_T\|^2_{L^2(\Omega)} \right], \tag{2.181}$$

where k is a positive number, arbitrarily large.

Both problems (2.179) and (2.181) admit a *unique* solution. There is however a technical difference between these two problems since problem (2.181) admits a unique solution under the only hypothesis $a_{ij} \in L^\infty(\Omega)$ (and of course the ellipticity property (2.171)), while the existence of a solution for problem (2.179), with β arbitrarily small, requires, so far, some regularity property (such as (2.175)) for the a_{ij}'s. In the following we shall assume that property (2.175) holds, even if this hypothesis is not always necessary.

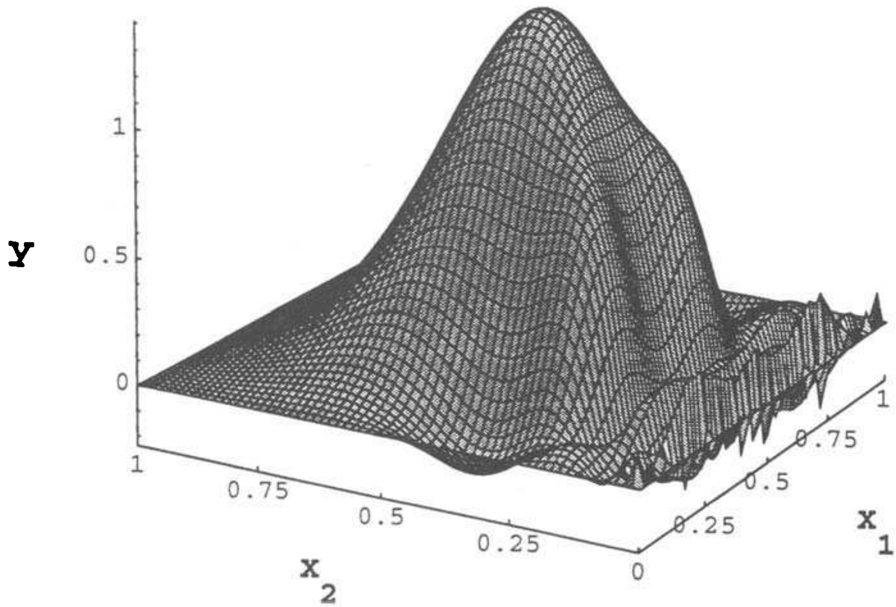


Fig. 29. Graph of the function $y_T^*(k = 10^7,\ h = \Delta t = 1/64)$.

2.8. Neumann control (II): Optimality conditions and dual formulations

The *optimality* system for problem (2.181) is obtained by arguments which are fairly classical (see, e.g., Lions (1968)), as recalled in Section 2.2. Following, precisely, the approach taken in Section 2.2, we introduce the functional $J_k : L^2(\Sigma_0) \to \mathbb{R}$ defined by

$$J_k(v) = \frac{1}{2}\int_{\Sigma_0} v^2\, \mathrm{d}\Sigma + \frac{1}{2}k\|y(T;v) - y_T\|^2_{L^2(\Omega)}. \tag{2.182}$$

We can show that the derivative J_k' of J_k is defined by

$$(J_k'(v), w)_{L^2(\Sigma_0)} = \int_{\Sigma_0} (v+p)w\, \mathrm{d}\Sigma, \ \forall v, w \in L^2(\Sigma_0), \tag{2.183}$$

where, in (2.183), the *adjoint state function* p is obtained from v via the solution of (2.166) and of the *adjoint state equation*

$$-\frac{\partial p}{\partial t} + A^* p = 0 \text{ in } Q, \quad \frac{\partial p}{\partial n_{A^*}} = 0 \text{ on } \Sigma,\ p(T) = k(y(T) - y_T). \tag{2.184}$$

Suppose now that u is *the* solution of the control problem (2.181); since $J_k'(u) = 0$, we have then the following optimality system satisfied by u and by the corresponding state and adjoint state functions:

$$u = -p|_{\Sigma_0}, \tag{2.185}$$

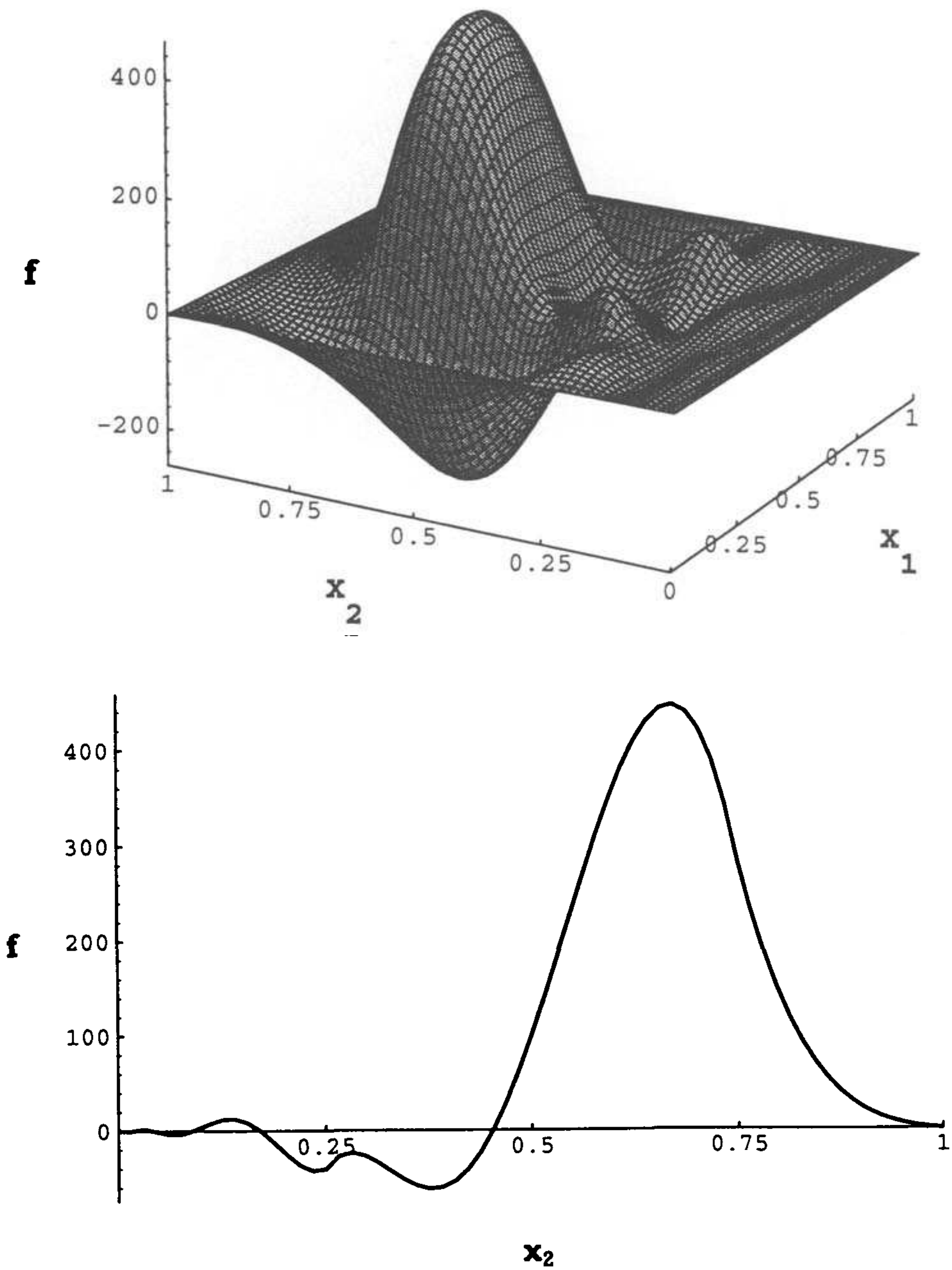


Fig. 30. (a) Graph of $f_h^{\Delta t}(k = 10^5,\ h = \Delta t = 1/64)$. (b) Graph of $x_2 \to f_h^{\Delta t}(0.5, x_2)(k = 10^5,\ h = \Delta t = 1/64)$.

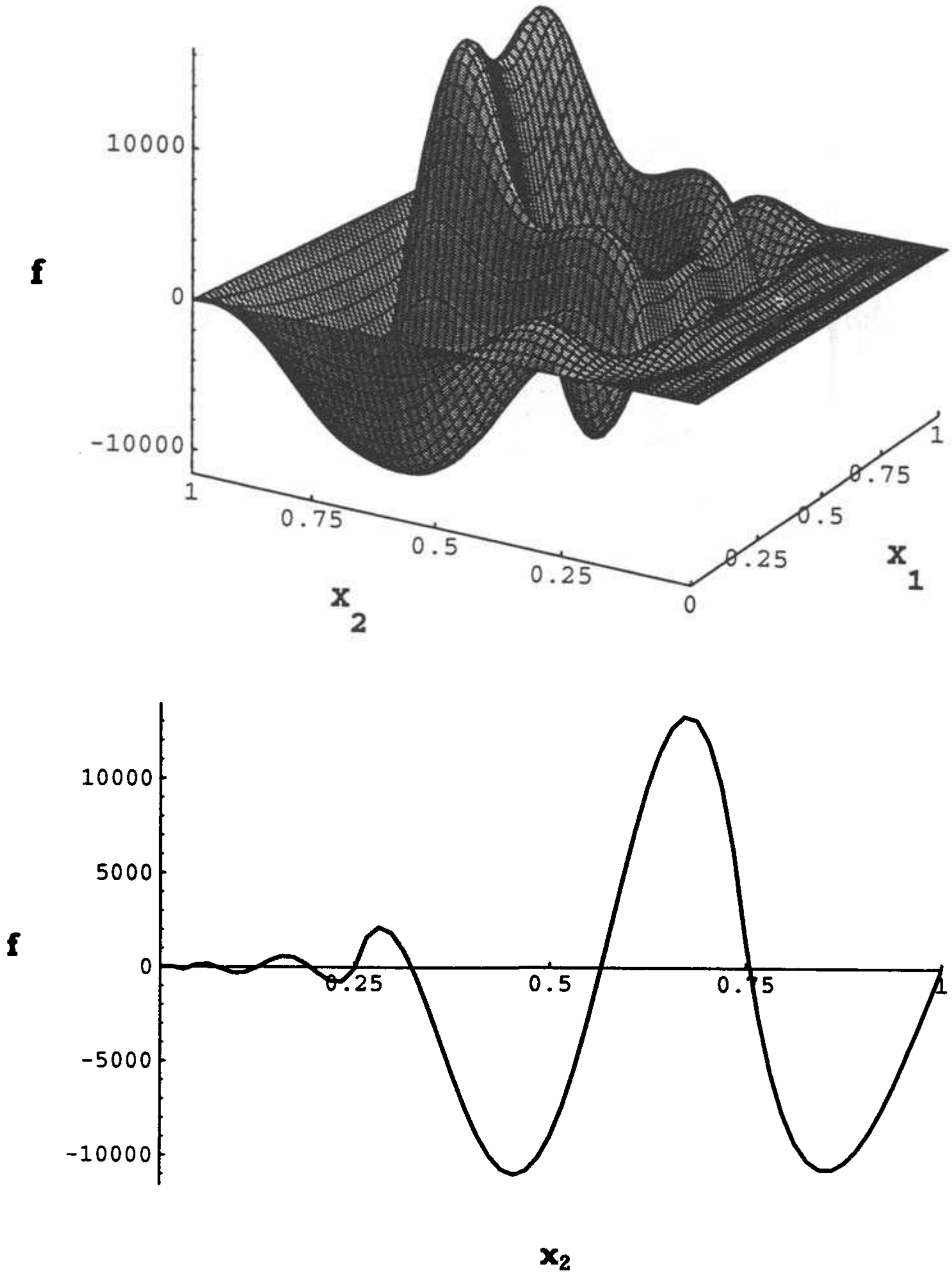


Fig. 31. (a) Graph of $f_h^{\Delta t}(k = 10^7,\ h = \Delta t = 1/64)$. (b) Graph of $x_2 \to f_h^{\Delta t}(0.5, x_2)(k = 10^7,\ h = \Delta t = 1/64)$.

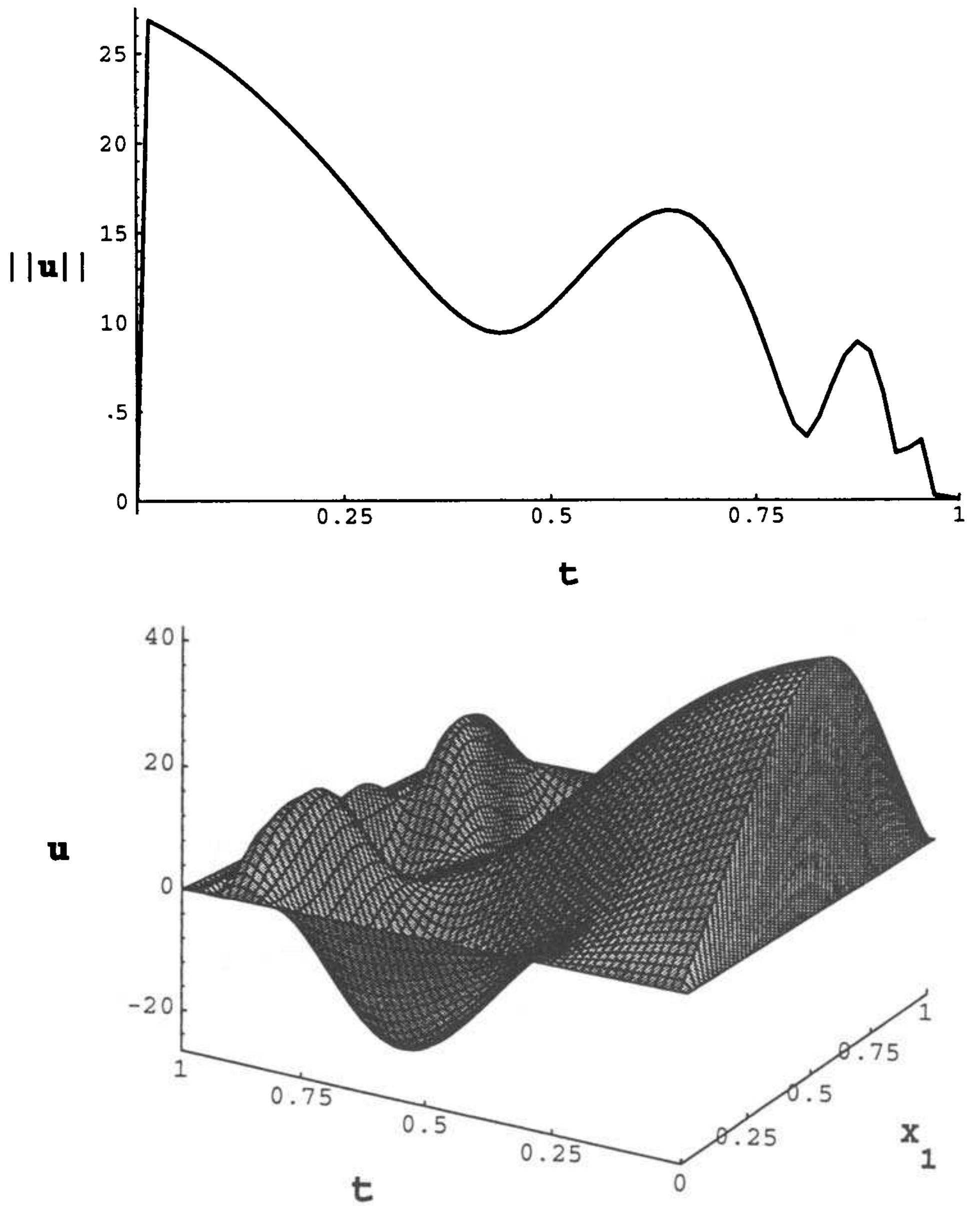
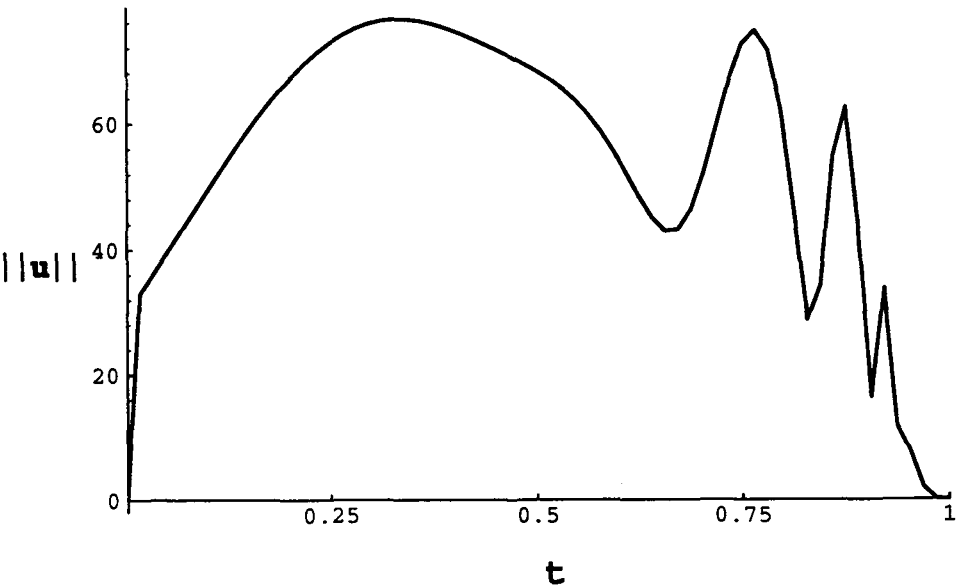


Fig. 32. (a) Graph of $t \to \|u^*(t)\|_{L^2(\Gamma_0)}(k = 10^5,\ h = \Delta t = 1/64)$. (b) Graph of the computed boundary control $(k = 10^5,\ h = \Delta t = 1/64)$.



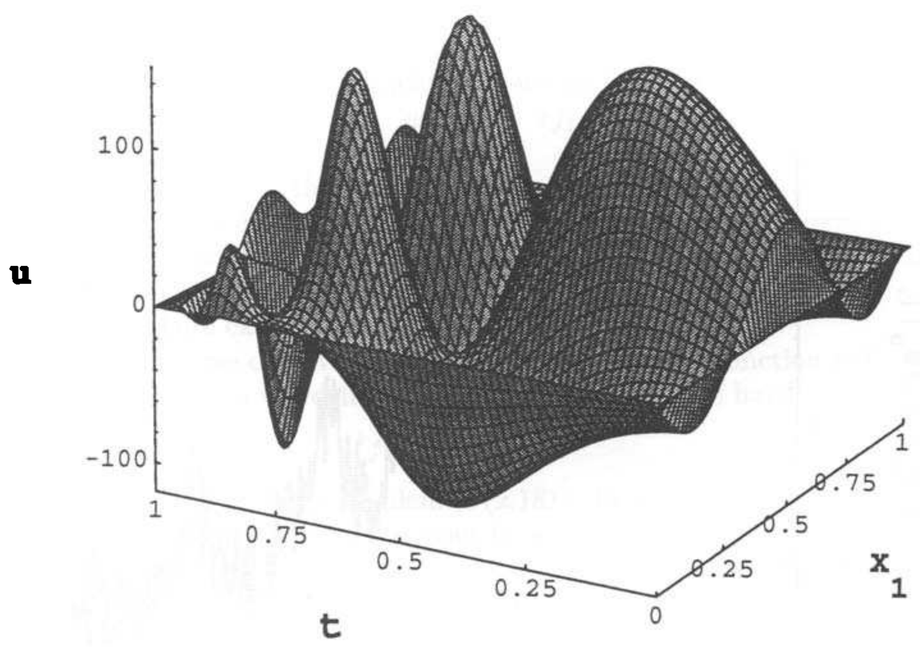


Fig. 33. (a) Graph of $t \to \|u^*(t)\|_{L^2(\Gamma_0)}(k = 10^7,\ h = \Delta t = 1/64)$. (b) Graph of the computed boundary control $(k = 10^7,\ h = \Delta t = 1/64)$.

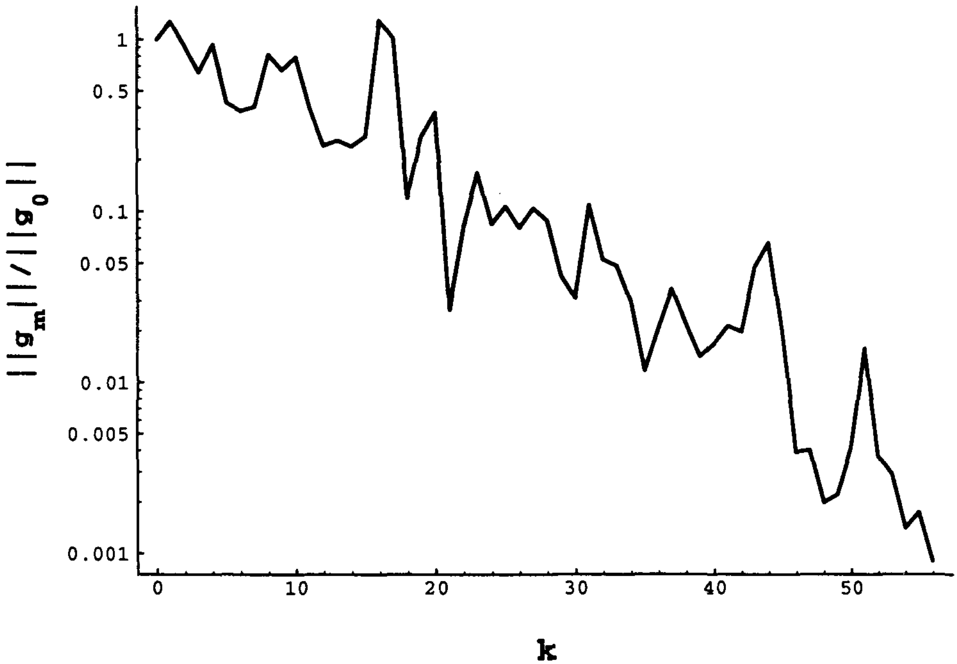


Fig. 34. Variation of $\|g_m\|_{H_0^1(\Omega)}/\|g_0\|_{H_0^1(\Omega)}(k = 10^5,\ h = \Delta t = 1/64)$.

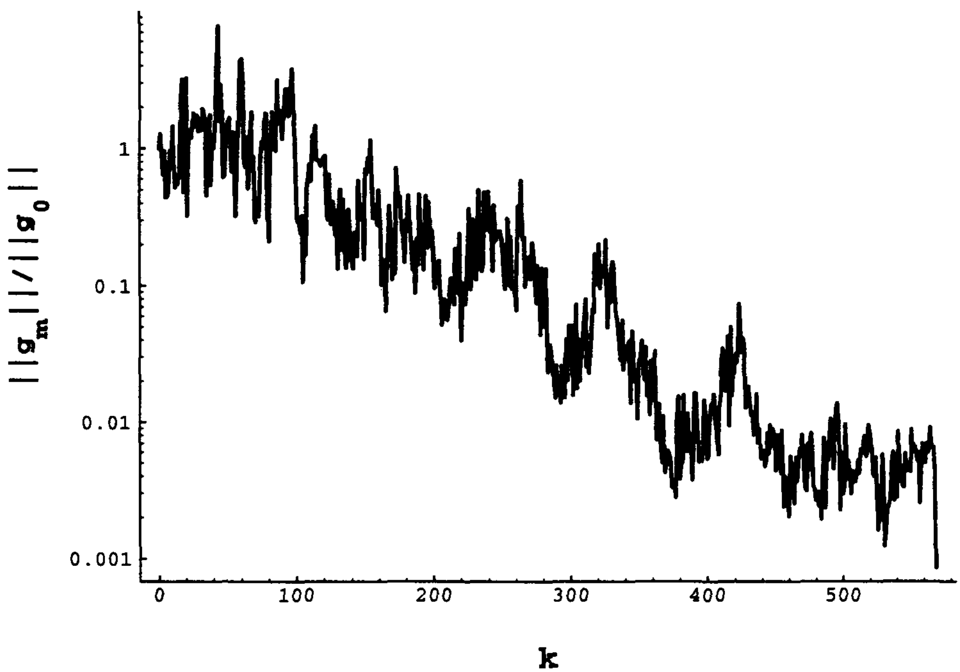


Fig. 35. Variation of $\|g_m\|_{H_0^1(\Omega)}/\|g_0\|_{H_0^1(\Omega)}(k = 10^7,\ h = \Delta t = 1/64)$.

$$\frac{\partial y}{\partial t} + Ay = 0 \text{ in } Q, \quad y(0) = 0, \; \frac{\partial y}{\partial n_A} = 0 \text{ on } \Sigma \backslash \Sigma_0, \; \frac{\partial y}{\partial n_A} = -p \text{ on } \Sigma_0, \tag{2.186}$$

$$-\frac{\partial p}{\partial t} + A^* p = 0 \text{ in } Q, \quad \frac{\partial p}{\partial n_{A^*}} = 0 \text{ on } \Sigma, \; p(T) = k(y(T) - y_T). \tag{2.187}$$

In order to identify the dual problem of (2.181) we proceed essentially as in Section 2.2. We introduce therefore the operator $\Lambda \in \mathcal{L}(L^2(\Omega); L^2(\Omega))$ defined by

$$\Lambda \hat{f} = -\hat{\varphi}(T), \; \forall \hat{f} \in L^2(\Omega), \tag{2.188}$$

where, in (2.188), $\hat{\varphi}$ is obtained from $\hat{f}$ as follows.

Solve first

$$-\frac{\partial \hat{\psi}}{\partial t} + A^* \hat{\psi} = 0 \text{ in } Q, \quad \frac{\partial \hat{\psi}}{\partial n_{A^*}} = 0 \text{ on } \Sigma, \; \hat{\psi}(T) = \hat{f}, \tag{2.189}$$

and then

$$\frac{\partial \hat{\varphi}}{\partial t} + A\hat{\varphi} = 0 \text{ in } Q, \quad \hat{\varphi}(0) = 0, \; \frac{\partial \hat{\varphi}}{\partial n_A} = 0 \text{ on } \Sigma \backslash \Sigma_0, \; \frac{\partial \hat{\varphi}}{\partial n_A} = -\hat{\psi} \text{ on } \Sigma_0. \tag{2.190}$$

We can easily show that (with obvious notation)

$$\int_\Omega (\Lambda f_1) f_2 \, \mathrm{d}x = \int_{\Sigma_0} \psi_1 \psi_2 \, \mathrm{d}\Sigma, \; \forall f_1, f_2 \in L^2(\Omega). \tag{2.191}$$

It follows from (2.191) that operator Λ is *symmetric* and *positive semi-definite*; indeed, it follows from the *Mizohata's uniqueness theorem* that operator Λ is *positive definite* (if (2.175) holds, at least). However, operator Λ is not an isomorphism from $L^2(\Omega)$ onto $L^2(\Omega)$ (implying that, in general, we do not have here exact boundary controllability).

Back to (2.188), we observe that, if we denote by f the function $p(T)$ in (2.187), it follows from the definition of operator Λ that we have

$$k^{-1} f + \Lambda f = -y_T. \tag{2.192}$$

Problem (2.192) is the dual problem of (2.181). From the properties of the operator $k^{-1}I + \Lambda$, problem (2.192) can be solved by a *conjugate gradient algorithm* operating in the space $L^2(\Omega)$; we shall return to this issue in Section 2.9.

The dual problem (2.192) has been obtained by a fairly simple method. Obtaining the dual problem of (2.179) is more complicated. We can use – as already done in previous sections – the *Fenchel–Rockafellar duality theory*; however, in order to introduce (possibly) our readers to other duality techniques we shall derive the dual problem of (2.179) through a *Lagrangian* approach (which is indeed closely related to the Fenchel–Rockafellar method, as shown in, e.g., Rockafellar (1970) and Ekeland and Temam (1974)).

Our starting point is to observe that problem (2.179) is equivalent to

$$\inf_{\{v,z\}} \frac{1}{2}\int_{\Sigma_0} v^2\,\mathrm{d}\Sigma, \tag{2.193}$$

where, in (2.193), the pair $\{v, z\}$ satisfies

$$v \in L^2(\Sigma_0), \tag{2.194}$$

$$z \in y_T + \beta B, \tag{2.195}$$

$$y(T) - z = 0, \tag{2.196}$$

$y(T)$ being obtained from v via the solution of (2.166). The idea here is to 'dualize' the linear constraint (2.196) via an appropriate *Lagrangian functional* and then to compute the corresponding *dual functional*. A Lagrangian functional naturally associated with problem (2.193)–(2.196) is defined by

$$\mathcal{L}(v, z; \mu) = \frac{1}{2}\int_{\Sigma_0} v^2\,\mathrm{d}\Sigma + \int_\Omega \mu(y(T) - z)\,\mathrm{d}x. \tag{2.197}$$

The dual problem associated with (2.193)–(2.197) is defined by

$$\inf_{\mu\in L^2(\Omega)} J^*(\mu), \tag{2.198}$$

where, in (2.198), the dual functional J^* is defined by

$$J^*(\mu) = -\inf_{\{v,z\}} \mathcal{L}(v, z; \mu), \tag{2.199}$$

where $\{v, z\}$ still satisfies (2.194), (2.195). We clearly have

$$\inf_{\{v,z\}} \mathcal{L}(v, z; \mu) = \inf_{v\in L^2(\Sigma_0)} \left[\frac{1}{2}\int_{\Sigma_0} v^2\,\mathrm{d}\Sigma + \int_\Omega y(T)\mu\,\mathrm{d}x\right] - \sup_{z\in y_T+\beta B}\int_\Omega \mu z\,\mathrm{d}x, \tag{2.200}$$

and then

$$\begin{aligned}\sup_{z\in y_T+\beta B}\int_\Omega \mu z\,\mathrm{d}x &= \sup_{z\in y_T+\beta B}\left[\int_\Omega \mu(z - y_T)\,\mathrm{d}x + \int_\Omega \mu y_T\,\mathrm{d}x\right] \\ &= \beta\|\mu\|_{L^2(\Omega)} + \int_\Omega \mu y_T\,\mathrm{d}x.\end{aligned} \tag{2.201}$$

It remains to evaluate

$$\inf_{v\in L^2(\Sigma_0)} \left[\frac{1}{2}\int_{\Sigma_0} v^2\,\mathrm{d}\Sigma + \int_\Omega y(T)\mu\,\mathrm{d}x\right]; \tag{2.202}$$

indeed, solving the (linear) control problem (2.202) is quite easy since its unique solution u_μ is characterized (see, e.g., Lions (1968)) by the existence of $\{y_\mu, p_\mu\}$ such that

$$u_\mu = -p_\mu|_{\Sigma_0}, \tag{2.203}$$

$$\frac{\partial y_\mu}{\partial t} + Ay_\mu = 0 \text{ in } Q, \quad y_\mu(0) = 0, \; \frac{\partial y_\mu}{\partial n_A} = -p_\mu \text{ on } \Sigma_0, \; \frac{\partial y_\mu}{\partial n_A} = 0 \text{ on } \Sigma\backslash\Sigma_0, \tag{2.204}$$

$$-\frac{\partial p_\mu}{\partial t} + A^* p_\mu = 0 \text{ in } Q, \quad \frac{\partial p_\mu}{\partial n_{A^*}} = 0 \text{ on } \Sigma, \; p_\mu(T) = \mu. \tag{2.205}$$

We have then from (2.202)–(2.205) and from the definition and properties of the operator Λ

$$\begin{aligned} \inf_{v\in L^2(\Sigma_0)} \left[\frac{1}{2}\int_{\Sigma_0} v^2 \, d\Sigma + \int_\Omega y(T)\mu \, dx\right] &= \frac{1}{2}\int_{\Sigma_0} p_\mu^2 \, d\Sigma + \int_\Omega y_\mu(T)\mu \, dx \\ &= -\frac{1}{2}\int_\Omega (\Lambda\mu)\mu \, dx. \end{aligned} \tag{2.206}$$

Combining (2.199), (2.200), (2.201) to (2.206) implies that

$$J^*(\mu) = \frac{1}{2}\int_\Omega (\Lambda\mu)\mu \, dx + \beta\|\mu\|_{L^2(\Omega)} + \int_\Omega y_T \mu \, dx. \tag{2.207}$$

The dual problem to (2.179) is defined then by

$$\inf_{\hat{f}\in L^2(\Omega)} \left[\frac{1}{2}\int_\Omega (\Lambda\hat{f})\hat{f} \, dx + \beta\|\hat{f}\|_{L^2(\Omega)} + \int_\Omega y_T \hat{f} \, dx\right], \tag{2.208}$$

or, equivalently, by the following *variational inequality*

$$\begin{cases} f \in L^2(\Omega), \\ \displaystyle\int_\Omega (\Lambda f)(\hat{f} - f) \, dx + \beta\|\hat{f}\|_{L^2(\Omega)} - \beta\|f\|_{L^2(\Omega)} \\ \qquad + \displaystyle\int_\Omega y_T(\hat{f} - f) \, dx \geq 0, \; \forall \hat{f} \in L^2(\Omega). \end{cases} \tag{2.209}$$

Once f is known, obtaining the solution u of problem (2.179) is quite easy, since

$$u = -p|_{\Sigma_0}, \tag{2.210}$$

where, in (2.210), p is the solution of

$$-\frac{\partial p}{\partial t} + A^* p = Q, \quad \frac{\partial p}{\partial n_{A^*}} = 0 \text{ on } \Sigma, \; p(T) = f. \tag{2.211}$$

The numerical solution of problem (2.208), (2.209) will be discussed in Section 2.10.

Remark 2.9 Proving *directly* the existence and uniqueness of the solution f of problem (2.208), (2.209) is not obvious. Actually, proving it without some *regularity* hypothesis on the a_{ij}'s (like (2.175)) is still an *open* question.

2.9. Neumann control (III): Conjugate gradient solution of the dual problem (2.192)

We shall address in this section the *iterative solution* of the control problem (2.181), via the solution of its *dual problem* (2.192). From the properties of Λ (*symmetry* and *positive definiteness*) problem (2.192) can be solved by a *conjugate gradient algorithm* operating in the space $L^2(\Omega)$. Such an algorithm is given below; we will use there a *variational description* in order to facilitate *finite element implementations* of the algorithm.

Description of the algorithm

$$f^0 \text{ is given in } L^2(\Omega); \tag{2.212}$$

solve

$$\begin{cases} -\int_\Omega \frac{\partial \psi^0}{\partial t}(t) z \,\mathrm{d}x + a(z, \psi^0(t)) = 0, \ \forall z \in H^1(\Omega), \\ \psi^0(t) \in H^1(\Omega), \quad \text{a.e. on } (0,T), \end{cases} \tag{2.213$_1$}$$

$$\psi^0(T) = f^0, \tag{2.213$_2$}$$

and then

$$\begin{cases} \int_\Omega \frac{\partial \varphi^0}{\partial t}(t) z \,\mathrm{d}x + a(\varphi^0(t), z) = -\int_{\Gamma_0} \psi^0(t) z \,\mathrm{d}\Gamma, \ \forall z \in H^1(\Omega), \\ \varphi^0(t) \in H^1(\Omega), \quad \text{a.e. on } (0,T), \end{cases} \tag{2.214$_1$}$$

$$\varphi^0(0) = 0. \tag{2.214$_2$}$$

Solve next

$$\begin{cases} g^0 \in L^2(\Omega), \\ \int_\Omega g^0 v \,\mathrm{d}x = k^{-1} \int_\Omega f^0 v \,\mathrm{d}x + \int_\Omega (y_T - \varphi^0(T)) v \,\mathrm{d}x, \ \forall v \in L^2(\Omega), \end{cases} \tag{2.215}$$

and set

$$w^0 = g^0. \tag{2.216}$$

Then, for $n \geq 0$, *assuming that* f^n, g^n, w^n *are known, compute* f^{n+1}, g^{n+1}, w^{n+1} *as follows.*
Solve

$$\begin{cases} -\int_\Omega \frac{\partial \bar{\psi}^n}{\partial t}(t) z \,\mathrm{d}x + a(z, \bar{\psi}^n(t)) = 0, \ \forall z \in H^1(\Omega), \\ \bar{\psi}^n(t) \in H^1(\Omega), \quad \text{a.e. on } (0,T) \quad , \end{cases} \tag{2.217$_1$}$$

$$\bar{\psi}^n(T) = w^n, \tag{2.217$_2$}$$

and then

$$\begin{cases} \int_\Omega \frac{\partial \bar{\varphi}^n}{\partial t}(t) z \, dx + a(\bar{\varphi}^n(t), z) = -\int_{\Gamma_0} \bar{\psi}^n(t) z \, d\Gamma, \ \forall z \in H^1(\Omega), \\ \bar{\varphi}^n(t) \in H^1(\Omega), \quad \text{a.e. on } (0,T) \quad , \end{cases} \tag{2.218$_1$}$$

$$\bar{\varphi}^n(0) = 0. \tag{2.218$_2$}$$

Solve next

$$\begin{cases} \bar{g}^n \in L^2(\Omega), \\ \int_\Omega \bar{g}^n v \, dx = k^{-1} \int_\Omega w^n v \, dx - \int_\Omega \bar{\varphi}^n(T) v \, dx, \ \forall v \in L^2(\Omega), \end{cases} \tag{2.219}$$

and compute

$$\rho_n = \int_\Omega |g^n|^2 \, dx \Big/ \int_\Omega \bar{g}^n w^n \, dx. \tag{2.220}$$

Set then

$$f^{n+1} = f^n - \rho_n w^n, \tag{2.221}$$

$$g^{n+1} = g^n - \rho_n \bar{g}^n. \tag{2.222}$$

If $\|g^{n+1}\|_{L^2(\Omega)}/\|g^0\|_{L^2(\Omega)} \leq \epsilon$, *take* $f = f^{n+1}$; *else, compute*

$$\gamma_n = \int_\Omega |g^{n+1}|^2 \, dx \Big/ \int_\Omega |g^n|^2 \, dx \tag{2.223}$$

and update w^{n+1} *via*

$$w^{n+1} = g^{n+1} + \gamma_n w^n. \tag{2.224}$$

Do $n = n + 1$ *and go to* (2.217).

In (2.212)–(2.224), the bilinear form $a(\cdot,\cdot)$ is defined by (2.173).

It is fairly easy to derive a fully discrete analogue of algorithm (2.212)–(2.224), obtained by combining finite elements for the space discretization and finite differences for the time discretization. We shall then obtain a variation of algorithm (2.125)–(2.151) (see Section 2.5), which is itself the fully discrete analogue of algorithm (2.42)–(2.54) (see Section 2.3). Actually, algorithm (2.212)–(2.224) is easier to implement than (2.42)–(2.54) since it operates in $L^2(\Omega)$, instead of $H_0^1(\Omega)$; no preconditioning is required, thus.

2.10. Neumann control (IV): Iterative solution of the dual problem (2.208), (2.209)

Problem (2.208), (2.209) can also be formulated as

$$-y_T \in \Lambda f + \beta \partial j(f), \tag{2.225}$$

which is a *multivalued* equation in $L^2(\Omega)$. In (2.225), $\partial j(\cdot)$ is the subgradient of the convex functional $j(\cdot)$ defined by

$$j(\hat{f}) = \left(\int_\Omega |\hat{f}|^2 \, \mathrm{d}x \right)^{1/2}, \ \forall \hat{f} \in L^2(\Omega).$$

As done in preceding sections we associate with the *elliptic equation* (2.225) the *initial value problem*

$$\begin{cases} \dfrac{\partial f}{\partial \tau} + \Lambda f + \beta \partial j(f) = -y_T, \\ f(0) = f_0. \end{cases} \tag{2.226}$$

To obtain the steady state solution of (2.226), i.e. the solution of (2.225), we shall use the following algorithm obtained, from (2.226), by application of the *Peaceman–Rachford time discretization scheme* (where $\Delta\tau(> 0)$ is a pseudo-time discretization step)

$$f^0 = f_0; \tag{2.227}$$

then, for $n \geq 0$, *compute* $f^{n+1/2}$ *and* f^{n+1}, *from* f^n, *via*

$$\frac{f^{n+1/2} - f^n}{\Delta\tau/2} + \Lambda f^n + \beta \partial j(f^{n+1/2}) = -y_T, \tag{2.228}$$

$$\frac{f^{n+1} - f^{n+1/2}}{\Delta\tau/2} + \Lambda f^{n+1} + \beta \partial j(f^{n+1/2}) = -y_T. \tag{2.229}$$

Problem (2.229) can be reformulated as

$$\frac{f^{n+1} - 2f^{n+1/2} + f^n}{\Delta\tau/2} + \Lambda f^{n+1} = \Lambda f^n; \tag{2.230}$$

problem (2.230) being a simple variation of problem (2.192) can be solved by an algorithm similar to (2.212)–(2.224). On the other hand, problem (2.228) can be (easily) solved by the methods used in Section 1.8.8 to solve problems (1.240), (1.243), (1.245) which are simple variants of problems (2.228).

3. CONTROL OF THE STOKES SYSTEM

3.1. Generalities. Synopsis

The control problems and methods which were discussed in Section 2 were mostly concerned with systems governed by *linear diffusion* equations of the *parabolic type*, associated with *second-order elliptic operators*. Indeed, these methods have been applied, in, e.g., Berggren (1992) and Berggren and Glowinski (1994), to the solution of *approximate boundary controllability* problems for systems governed by *strongly* advection dominated *linear* *advection–diffusion equations.* These methods can also be applied to *systems* of linear advection–diffusion equations and to *higher-order parabolic equations* (or *systems* of such equations). Motivated by the solution of controllability problems for the *Navier–Stokes equations* modelling *incompressible* viscous flow, we will now discuss controllability issues for a system of partial differential equations *which is not of the Cauchy–Kowalewski type*, namely the classical *Stokes* system.

3.2. Formulation of the Stokes system. A fundamental controllability result

In the following, we equip the Euclidian space $\mathbb{R}^d (d \geq 2)$ with its classical scalar product and with the corresponding norm, i.e.

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{d} a_i b_i, \ \forall \mathbf{a} = \{a_i\}_{i=1}^{d}, \ \mathbf{b} = \{b_i\}_{i=1}^{d} \in \mathbb{R}^d; \ |\mathbf{a}| = (\mathbf{a} \cdot \mathbf{a})^{1/2}, \ \forall \mathbf{a} \in \mathbb{R}^d.$$

We suppose from now on that the control $\mathbf{v}$ is distributed over Ω, with its support in $\bar{\mathcal{O}} \subset \Omega$ (as in Sections 1.1 to 1.8, whose notation is kept). The *state equation* is given by

$$\begin{cases} \dfrac{\partial \mathbf{y}}{\partial t} - \Delta \mathbf{y} = \mathbf{v}\chi_{\mathcal{O}} - \boldsymbol{\nabla}\pi \text{ in } Q, \\ \boldsymbol{\nabla} \cdot \mathbf{y} = 0 \text{ in } Q, \end{cases} \tag{3.1}$$

subjected to the following *initial* and *boundary* conditions

$$\mathbf{y}(0) = \mathbf{0}, \quad \mathbf{y} = \mathbf{0} \text{ on } \Sigma. \tag{3.2}$$

In (3.1) we shall assume that

$$\mathbf{v} \in \mathcal{V} = \text{ closed subpsace of } L^2(\mathcal{O} \times (0,T))^d. \tag{3.3}$$

To fix ideas we shall take $d = 3$, and consider the following cases for $\mathcal{V}$:

$$\mathcal{V} = L^2(\mathcal{O} \times (0,T))^3, \tag{3.4}$$

$$\mathcal{V} = \{v_1, v_2, 0\}, \quad \{v_1, v_2\} \in L^2(\mathcal{O} \times (0,T))^2, \tag{3.5}$$

$$\mathcal{V} = \{v_1, 0, 0\}, \quad v_1 \in L^2(\mathcal{O} \times (0,T)). \tag{3.6}$$

Problem (3.1), (3.2) has a *unique* solution, such that (in particular)

$$\begin{cases} \mathbf{y}(t;\mathbf{v}) \in L^2(0,T;(H_0^1(\Omega))^3), \ \boldsymbol{\nabla} \cdot \mathbf{y} = 0, \\ \dfrac{\partial \mathbf{y}}{\partial t}(t;\mathbf{v}) \in L^2(0,T;V'), \end{cases} \tag{3.7}$$

where V' is the dual space of V with

$$V = \{\boldsymbol{\varphi} \mid \boldsymbol{\varphi} \in (H_0^1(\Omega))^3, \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} = 0\}. \tag{3.8}$$

It follows from (3.7) that

$$t \to \mathbf{y}(t; \mathbf{v}) \text{ belongs to } C^0([0,T]; H), \tag{3.9}$$

where

$$\begin{aligned} H = & \text{ closure of } V \text{ in } (L^2(\Omega))^3 \\ & = \{\boldsymbol{\varphi} \mid \boldsymbol{\varphi} \in (L^2(\Omega))^3, \ \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} = 0, \ \boldsymbol{\varphi} \cdot \mathbf{n} = 0 \text{ on } \Gamma\} \end{aligned} \tag{3.10}$$

(where $\mathbf{n}$ denotes the outward unit normal vector at Γ).

We are now going to prove the following

Proposition 3.1 *If $\mathcal{V}$ is defined by either* (3.4) *or* (3.5), *then the space spanned by $\mathbf{y}(T; \mathbf{v})$ is dense in H.*

Proof. It suffices to prove the above results for the case (3.5). Let us therefore consider $\mathbf{f} \in H$ such that

$$\int_\Omega \mathbf{y}(T; \mathbf{v}) \cdot \mathbf{f} \, dx = 0, \ \forall \mathbf{v} \in \mathcal{V}. \tag{3.11}$$

To $\mathbf{f}$ we associate the solution $\boldsymbol{\psi}$ of the following *backward* Stokes problem

$$\begin{cases} -\dfrac{\partial \boldsymbol{\psi}}{\partial t} - \Delta \boldsymbol{\psi} = -\boldsymbol{\nabla} \sigma \text{ in } Q, \\ \boldsymbol{\nabla} \cdot \boldsymbol{\psi} = 0 \text{ in } Q, \end{cases} \tag{3.12}$$

$$\boldsymbol{\psi}(T) = \mathbf{f}, \ \boldsymbol{\psi} = \mathbf{0} \text{ on } \Sigma. \tag{3.13}$$

Multiplying by $\mathbf{y} = \mathbf{y}(\mathbf{v})$ the first equation in (3.12) and integrating by parts we find that

$$\iint_{\mathcal{O} \times (0,T)} \boldsymbol{\psi} \cdot \mathbf{v} \, dx \, dt = 0, \ \forall \mathbf{v} \in \mathcal{V}. \tag{3.14}$$

Therefore

$$\psi_1 = \psi_2 = 0 \text{ in } \mathcal{O} \times (0,T). \tag{3.15}$$

But $\boldsymbol{\psi}$ is (among other things) *continuous* in t and *real analytic* in x in $\Omega \times (0,T)$, so that (3.15) implies that

$$\psi_1 = \psi_2 = 0 \text{ in } \Omega \times (0,T). \tag{3.16}$$

Since $\boldsymbol{\nabla} \cdot \boldsymbol{\psi} = 0$, it follows from (3.16) that $\partial \psi_3 / \partial x_3 = 0$ in $\Omega \times (0,T)$, and since $\psi_3 = 0$ on Σ, then $\psi_3 = 0$ in $\Omega \times (0,T)$, so that $\mathbf{f} = \mathbf{0}$, which completes the proof. □

Remark 3.1 The above density result does not always hold if $\mathcal{V}$ is defined by (3.6), as proven by I. Diaz and Fursikov (1994).

Remark 3.2 Proposition 3.1 was proved in the lectures of the second author at *Collège de France* in 1990/91. Other results along these lines are due to Fursikov (1992).

The *density* result in Proposition 3.1 implies (at least) *approximate controllability.* Thus, we shall formulate and discuss, in the following sections, two approximate controllability problems.

3.3. Two approximate controllability problems

The *first problem* is defined by

$$\min_{\mathbf{v}\in\mathcal{U}_f} \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |\mathbf{v}|^2 \, dx \, dt, \tag{3.17}$$

where

$$\mathcal{U}_f = \{\mathbf{v} \mid \mathbf{v} \in \mathcal{V}, \{\mathbf{v},\mathbf{y}\} \text{ verifies } (3.1), (3.2) \text{ and } \mathbf{y}(T) \in \mathbf{y}_T + \beta B_H\}; \tag{3.18}$$

in (3.18), $\mathbf{y}_T$ is given in H, β is an arbitrary small positive number, B_H is the closed unit ball of H and – to fix ideas – $\mathcal{V}$ is defined by (3.5).

The *second problem* is obtained by *penalization* of the final condition $\mathbf{y}(T) = \mathbf{y}_T$; we have then

$$\min_{\mathbf{v}\in\mathcal{V}} \left[\frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |\mathbf{v}|^2 \, dx \, dt + \frac{1}{2}k\int_\Omega |\mathbf{y}(T) - \mathbf{y}_T|^2 \, dx\right], \tag{3.19}$$

where, in (3.19), k is an arbitrary large positive number, $\mathbf{y}$ is obtained from $\mathbf{v}$ via (3.1), (3.2) and $\mathcal{V}$ is as above.

It follows from Proposition 3.1 that both control problems (3.17) and (3.19) *have a unique solution.*

3.4. Optimality conditions and dual problems

We start with problem (3.19), since it is simpler than problem (3.17). If we denote by J_k the cost functional in (3.19), we have

$$\lim_{\substack{\theta\to 0\\ \theta\neq 0}} \frac{J_k(\mathbf{v}+\theta\mathbf{w}) - J_k(\mathbf{v})}{\theta} = (J_k'(\mathbf{v}),\mathbf{w}) = \iint_{\mathcal{O}\times(0,T)} (\mathbf{v}-\mathbf{p})\cdot\mathbf{w} \, dx \, dt, \tag{3.20}$$

where, in (3.20), the *adjoint velocity field* $\mathbf{p}$ is solution of the following *backward Stokes problem*

$$\begin{cases} -\dfrac{\partial \mathbf{p}}{\partial t} - \Delta\mathbf{p} + \nabla\sigma = \mathbf{0} \text{ in } Q, \\ \nabla\cdot\mathbf{p} = 0 \text{ in } Q, \end{cases} \tag{3.21}$$

$$\mathbf{p} = \mathbf{0} \text{ on } \Sigma, \; \mathbf{p}(T) = k(\mathbf{y}_T - \mathbf{y}(T)). \tag{3.22}$$

Suppose now that $\mathbf{u}$ is the unique solution of problem (3.19); it is *characterized* by

$$\begin{cases} \mathbf{u} \in \mathcal{V}, \\ (J_k'(\mathbf{u}),\mathbf{w}) = 0, \; \forall \mathbf{w} \in \mathcal{V}, \end{cases} \tag{3.23}$$

which implies in turn that the *optimal triple* $\{\mathbf{u}, \mathbf{y}, \mathbf{p}\}$ is characterized by

$$u_1 = p_1|_{\mathcal{O}},\ u_2 = p_2|_{\mathcal{O}},\ u_3 = 0, \tag{3.24}$$

$$\begin{cases} \dfrac{\partial \mathbf{y}}{\partial t} - \Delta \mathbf{y} + \boldsymbol{\nabla}\pi = \mathbf{u}\chi_{\mathcal{O}} \text{ in } Q, \\ \boldsymbol{\nabla} \cdot \mathbf{y} = 0 \text{ in } Q, \end{cases} \tag{3.25}$$

$$\mathbf{y}(0) = \mathbf{0},\ \mathbf{y} = \mathbf{0} \text{ on } \Sigma, \tag{3.26}$$

to be completed by (3.21), (3.22).

To obtain the *dual problem* of (3.19) from the above optimality conditions we proceed as in the preceding sections by introducing an operator $\Lambda \in \mathcal{L}(H; H)$ defined as follows:

$$\Lambda \hat{\mathbf{f}} = \hat{\boldsymbol{\varphi}}(T),\ \forall \hat{\mathbf{f}} \in H, \tag{3.27}$$

where to obtain $\hat{\boldsymbol{\varphi}}(T)$ we solve first

$$\begin{cases} -\dfrac{\partial \hat{\boldsymbol{\psi}}}{\partial t} - \Delta \hat{\boldsymbol{\psi}} + \boldsymbol{\nabla}\hat{\sigma} = \mathbf{0} \text{ in } Q, \\ \boldsymbol{\nabla} \cdot \hat{\boldsymbol{\psi}} = 0 \text{ in } Q, \end{cases} \tag{3.28}$$

$$\hat{\boldsymbol{\psi}}(T) = \hat{\mathbf{f}},\ \hat{\boldsymbol{\psi}} = \mathbf{0} \text{ on } \Sigma, \tag{3.29}$$

and then

$$\begin{cases} \dfrac{\partial \hat{\boldsymbol{\varphi}}}{\partial t} - \Delta \hat{\boldsymbol{\varphi}} + \boldsymbol{\nabla}\hat{\pi} = \{\hat{\psi}_1, \hat{\psi}_2, 0\}\chi_{\mathcal{O}} \text{ in } Q, \\ \boldsymbol{\nabla} \cdot \hat{\boldsymbol{\varphi}} = 0 \text{ in } Q, \end{cases} \tag{3.30}$$

$$\hat{\boldsymbol{\varphi}}(0) = \mathbf{0},\ \hat{\boldsymbol{\varphi}} = \mathbf{0} \text{ on } \Sigma \tag{3.31}$$

(the two above *Stokes problems* are *well-posed*).

Integrating by parts in time and using Green's formula we can show that (with obvious notation) we have

$$\int_\Omega (\Lambda \hat{\mathbf{f}}) \cdot \hat{\mathbf{f}}' \, \mathrm{d}x = \iint_{\mathcal{O}\times(0,T)} (\hat{\psi}_1 \hat{\psi}_1' + \hat{\psi}_2 \hat{\psi}_2') \, \mathrm{d}x \, \mathrm{d}t,\ \forall \hat{f},\ \hat{f}' \in H. \tag{3.32}$$

It follows from relation (3.32) that the operator Λ is *symmetric* and *positive semi-definite* over H; indeed using the approach taken in Section 3.2 to prove Proposition 3.1, we can show that Λ is *positive definite* over H.

Back to the *optimality conditions*, let us denote by $\mathbf{f}$ the function $\mathbf{p}(T)$; it follows then from (3.22) and from the definition of Λ that $\mathbf{f}$ satisfies

$$k^{-1}\mathbf{f} + \Lambda \mathbf{f} = \mathbf{y}_T \tag{3.33}$$

which is precisely the dual problem of (3.19).

From the symmetry of Λ, problem (3.33) can be solved by a *conjugate gradient algorithm* operating in the space H.

Consider control problem (3.17); applying the *Fenchel–Rockafellar duality theory* it can be shown that the unique solution $\mathbf{u}$ of problem (3.17) can be obtained via

$$u_1 = p_1\chi_{\mathcal{O}},\ u_2 = p_2\chi_{\mathcal{O}},\ u_3 = 0, \tag{3.34}$$

where, in (3.34), $\mathbf{p}$ is the solution of the backward Stokes problem

$$\begin{cases} -\dfrac{\partial \mathbf{p}}{\partial t} - \Delta\mathbf{p} + \boldsymbol{\nabla}\sigma = \mathbf{0} \text{ in } Q, \\ \boldsymbol{\nabla}\cdot\mathbf{p} = 0 \text{ in } Q, \end{cases} \tag{3.35}$$

$$\mathbf{p}(T) = \mathbf{f},\ \mathbf{p} = \mathbf{0} \text{ on } \Sigma, \tag{3.36}$$

where, in (3.36), $\mathbf{f}$ is *the* solution of the following *variational inequality*

$$\begin{cases} \mathbf{f} \in H;\ \forall\hat{\mathbf{f}} \in H, \quad \text{we have} \\ \displaystyle\int_\Omega (\Lambda\mathbf{f})\cdot(\hat{\mathbf{f}} - \mathbf{f})\,dx + \beta\|\hat{\mathbf{f}}\|_H - \beta\|\mathbf{f}\|_H \geq \int_\Omega \mathbf{y}_T\cdot(\hat{\mathbf{f}} - \mathbf{f})\,dx \end{cases} \tag{3.37}$$

where $\|\mathbf{f}\|_H = (\int_\Omega |\mathbf{f}|^2\,dx)^{1/2}$.

Problem (3.37) can be viewed as the *dual* of problem (3.17).

3.5. Iterative solution of the control problem

The various *primal* or *dual* control problems considered in Sections 3.3 and 3.4 can be solved by variants of the algorithms which have been used to solve their scalar diffusion analogues; these algorithms have been described in Section 1.8. Here we shall focus on the *direct solution* of the control problem (3.19), by a *conjugate gradient algorithm*, since we used this approach to solve the test problem discussed in Section 3.7. The unique solution $\mathbf{u}$ of the control problem (3.19) is *characterized* as also being the unique solution of the linear variational problem (3.23). From the properties of the functional J_k, this problem is a particular case of problem (1.121) in Section 1.8.2; applying thus algorithm (1.122)–(1.129) to problem (3.23) we obtain:

$$\mathbf{u}^0 \quad \text{chosen in } \mathcal{V}; \tag{3.38}$$

solve

$$\begin{cases} \dfrac{\partial \mathbf{y}^0}{\partial t} - \Delta\mathbf{y}^0 + \boldsymbol{\nabla}\pi^0 = \mathbf{u}^0\chi_{\mathcal{O}} \text{ in } Q, \\ \boldsymbol{\nabla}\cdot\mathbf{y}_0 = 0 \text{ in } Q, \end{cases} \tag{3.39}$$

$$\mathbf{y}^0(0) = \mathbf{0},\ \mathbf{y}^0 = \mathbf{0} \text{ on } \Sigma, \tag{3.40}$$

and then

$$\begin{cases} -\dfrac{\partial \mathbf{p}^0}{\partial t} - \Delta\mathbf{p}^0 + \boldsymbol{\nabla}\sigma^0 = \mathbf{0} \text{ in } Q, \\ \boldsymbol{\nabla}\cdot\mathbf{p}^0 = 0 \text{ in } Q, \end{cases} \tag{3.41}$$

$$\mathbf{p}^0 = \mathbf{0} \text{ on } \Sigma,\ \mathbf{p}^0(T) = k(\mathbf{y}_T - \mathbf{y}^0(T)). \tag{3.42}$$

Solve now

$$\begin{cases} \mathbf{g}^0 \in \mathcal{V}, \\ \displaystyle\iint_{\mathcal{O}\times(0,T)} \mathbf{g}^0 \cdot \mathbf{v}\, dx\, dt = \iint_{\mathcal{O}\times(0,T)} (\mathbf{u}^0 - \mathbf{p}^0)\cdot \mathbf{v}\, dx\, dt,\ \forall \mathbf{v} \in \mathcal{V}, \end{cases} \tag{3.43}$$

and set

$$\mathbf{w}^0 = \mathbf{g}^0. \tag{3.44}$$

Then for $n \geq 0$, assuming that $\mathbf{u}^n$, $\mathbf{g}^n$, $\mathbf{w}^n$ are known, we obtain $\mathbf{u}^{n+1}$, $\mathbf{g}^{n+1}$, $\mathbf{w}^{n+1}$ as follows.

Solve

$$\begin{cases} \dfrac{\partial \bar{\mathbf{y}}^n}{\partial t} - \Delta \bar{\mathbf{y}}^n + \boldsymbol{\nabla} \bar{\pi}^n = \mathbf{w}^n \chi_{\mathcal{O}} \text{ in } Q, \\ \boldsymbol{\nabla} \cdot \bar{\mathbf{y}}^n = 0 \text{ in } Q, \end{cases} \tag{3.45}$$

$$\bar{\mathbf{y}}^n(0) = \mathbf{0},\ \bar{\mathbf{y}}^n = \mathbf{0} \text{ on } \Sigma, \tag{3.46}$$

and then

$$\begin{cases} -\dfrac{\partial \bar{\mathbf{p}}^n}{\partial t} - \Delta \bar{\mathbf{p}}^n + \boldsymbol{\nabla} \bar{\sigma}^n = \mathbf{0} \text{ in } Q, \\ \boldsymbol{\nabla} \cdot \bar{\mathbf{p}}^n = \mathbf{0} \text{ in } Q, \end{cases} \tag{3.47}$$

$$\bar{\mathbf{p}}^n = \mathbf{0} \text{ on } \Sigma,\ \bar{\mathbf{p}}^n(T) = -k\bar{\mathbf{y}}^n(T). \tag{3.48}$$

Solve now

$$\begin{cases} \bar{\mathbf{g}}^n \in \mathcal{V}, \\ \displaystyle\iint_{\mathcal{O}\times(0,T)} \bar{\mathbf{g}}^n \cdot \mathbf{v}\, dx\, dt = \iint_{\mathcal{O}\times(0,T)} (\bar{\mathbf{u}}^n - \bar{\mathbf{p}}^n)\cdot \mathbf{v}\, dx\, dt,\ \forall \mathbf{v} \in \mathcal{V}, \end{cases} \tag{3.49}$$

and compute

$$\rho_n = \iint_{\mathcal{O}\times(0,T)} |\mathbf{g}^n|^2\, dx\, dt \Big/ \iint_{\mathcal{O}\times(0,T)} \bar{\mathbf{g}}^n \cdot \mathbf{w}^n\, dx\, dt, \tag{3.50}$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n - \rho_n \mathbf{w}^n, \tag{3.51}$$

$$\mathbf{g}^{n+1} = \mathbf{g}^n - \rho_n \bar{\mathbf{g}}^n. \tag{3.52}$$

If $\|\mathbf{g}^{n+1}\|_{L^2(\mathcal{O}\times(0,T))^d} / \|\mathbf{g}^0\|_{L^2(\mathcal{O}\times(0,T))^d} \leq \epsilon$ take $\mathbf{u} = \mathbf{u}^{n+1}$; else, compute

$$\gamma_n = \iint_{\mathcal{O}\times(0,T)} |\mathbf{g}^{n+1}|^2\, dx\, dt \Big/ \iint_{\mathcal{O}\times(0,T)} |\mathbf{g}^n|^2\, dx\, dt \tag{3.53}$$

and update $\mathbf{w}^n$ by

$$\mathbf{w}^{n+1} = \mathbf{g}^{n+1} + \gamma_n \mathbf{w}^n. \tag{3.54}$$

Do $n = n+1$ and go to (3.45).

Remark 3.3 For a given value of ϵ the number of iterations necessary to obtain the convergence of algorithm (3.38)–(3.54) varies like $k^{1/2}$ (as before for closely related algorithms).

Remark 3.4 The implementation of algorithm (3.38)–(3.54) requires *efficient Stokes solvers*, for solving problems (3.39) (3.40), (3.41) (3.42), (3.45) (3.46), (3.47) and (3.48). Such solvers can be found in, e.g., Glowinski and Le Tallec (1989), Glowinski (1991; 1992a); actually, this issue is fully addressed in the related article by Berggren and Glowinski (1994), for *more general* boundary conditions than Dirichlet.

3.6. Time discretization of the control problem (3.19)

The practical implementation of algorithm (3.38)–(3.54) requires space and time approximations of the control problem (3.19). Focusing on the *time discretization* only (the space discretization will be addressed in Berggren and Glowinski (1994)) we introduce a *time discretization step* $\Delta t = T/N$ (with N a *positive* integer), denote by $\mathbf{v}$ the vector $\{\mathbf{v}^n\}_{n=1}^N$ and approximate problem (3.19) by

$$\min_{\mathbf{v}\in\mathcal{V}^{\Delta t}} \left[\frac{\Delta t}{2}\sum_{n=1}^{N}\int_{\mathcal{O}} |\mathbf{v}^n|^2\,\mathrm{d}x + \frac{k}{2}\int_\Omega |\mathbf{y}^N - \mathbf{y}_T|^2\,\mathrm{d}x\right], \tag{3.55}$$

where, by analogy with (3.4)–(3.6), $\mathcal{V}^{\Delta t}$ is defined by either

$$\mathcal{V}^{\Delta t} = \Big\{\{\mathbf{v}^n\}_{n=1}^N \mid \mathbf{v}^n = \{v_1^n, v_2^n, v_3^n\} \in (L^2(\mathcal{O}))^3,\ \forall n = 1,\ldots,N\Big\}$$

or

$$\mathcal{V}^{\Delta t} = \Big\{\{\mathbf{v}^n\}_{n=1}^N \mid \mathbf{v}^n = \{v_1^n, v_2^n, 0\}, \{v_1^n, v_2^n,\} \in (L^2(\mathcal{O}))^2,\ \forall n = 1,\ldots,N\Big\},$$

$$\mathcal{V}^{\Delta t} = \Big\{\{\mathbf{v}^n\}_{n=1}^N \mid \mathbf{v}^n = \{v_1^n, 0, 0\}, v_1^n \in L^2(\mathcal{O}),\ \forall n = 1,\ldots,N\Big\},$$

and where $\mathbf{y}^n$ is obtained from $\mathbf{v}$ via

$$\mathbf{y}^0 = \mathbf{0}; \tag{3.56}$$

for $n = 1,\ldots,N$, we obtain $\{\mathbf{y}^n, \pi^n\}$ from $\mathbf{y}^{n-1}$ by solving the following steady Stokes type problem

$$\begin{cases} \dfrac{\mathbf{y}^n - \mathbf{y}^{n-1}}{\Delta t} - \Delta\mathbf{y}^n + \boldsymbol{\nabla}\pi^n = \mathbf{v}^n\chi_{\mathcal{O}} \text{ in } \Omega, \\ \boldsymbol{\nabla}\cdot\mathbf{y}^n = 0 \text{ in } \Omega, \end{cases} \tag{3.57}$$

$$\mathbf{y}^n = \mathbf{0} \text{ on } \Gamma. \tag{3.58}$$

The above scheme is nothing but a *backward Euler time discretization* of problem (3.1), (3.2). Efficient algorithms for solving problem (3.57), (3.58) (and finite element approximations of it) can be found in, e.g., Glowinski and

Le Tallec (1989), Glowinski (1991; 1992a) (see also Berggren and Glowinski (1994)).

The discrete control problem (3.55) has a *unique* solution; for the optimality conditions and a discrete analogue of the conjugate gradient algorithm (3.38)–(3.54) see Berggren and Glowinski (1994) (see also the above reference for a discussion of the *full discretization* of problem (3.19) and solution methods for the fully discrete problem).

3.7. Numerical experiments

Following Berggren and Glowinski (1994), we (briefly) consider the practical solution of the following variant of problem (3.19):

$$\min_{\mathbf{v}\in\mathcal{V}} \left[\frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |\mathbf{v}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{k}{2}\int_\Omega |\mathbf{y}(T) - \mathbf{y}_T|^2 \,\mathrm{d}x\right], \tag{3.59}$$

where, in (3.59), $\mathcal{O} \subset \Omega \subset \mathbb{R}^2$, $\mathbf{v} = \{v_1, 0\}$, $\mathcal{V} = \{\mathbf{v} \mid \mathbf{v} = \{v_1, 0\}, v_1 \in L^2(\mathcal{O} \times (0,T))\}$, where $\mathbf{y}(T)$ is obtained from $\mathbf{v}$ via the solution of the following *Stokes problem*

$$\begin{cases} \dfrac{\partial \mathbf{y}}{\partial t} - \nu\Delta\mathbf{y} + \boldsymbol{\nabla}\pi = \mathbf{v}\chi_{\mathcal{O}} \text{ in } Q, \\ \boldsymbol{\nabla}\cdot\mathbf{y} = 0 \text{ in } Q, \end{cases} \tag{3.60}$$

$$\mathbf{y}(0) = \mathbf{y}_0, \text{ with } \mathbf{y}_0 \in (L^2(\Omega))^2,\ \boldsymbol{\nabla}\cdot\mathbf{y}_0 = 0,\ \mathbf{y}_0\cdot\mathbf{n} = 0 \text{ on } \Sigma_0(=\Gamma_0\times(0,T)), \tag{3.61}$$

$$\mathbf{y} = \mathbf{g}_0 \text{ on } \Sigma_0, \tag{3.62}$$

$$\nu\frac{\partial \mathbf{y}}{\partial n} - \mathbf{n}\pi = \mathbf{g}_1 \text{ on } \Sigma_1(=\Gamma_1\times(0,T)), \tag{3.63}$$

and where the target function $\mathbf{y}_T$ is given in $(L^2(\Omega))^2$. In (3.60)–(3.63) $\nu(>0)$ is a viscosity parameter and $\Gamma_0 \cap \Gamma_1 = \emptyset$, closure of $\Gamma_0 \cup \Gamma_1 = \Gamma$. Actually the boundary condition (3.63) is not particularly physical, but it can be used to implement downstream boundary conditions for flow in unbounded regions.

The test problem that we consider is the particular problem (3.59) where:

1 $\Omega = (0,2)\times(0,1)$, $\mathcal{O} = (1/2, 3/2)\times(1/4, 3/4)$, $T = 1$;
2 $\Gamma_0 = \{\{x_i\}_{i=1}^2 \mid x_2 = 0 \text{ or } 1,\ 0 < x_1 < 2\}$, $\Gamma_1 = \{\{x_i\}_{i=1}^2 \mid x_1 = 0 \text{ or } 2,\ 0 < x_2 < 1\}$;
3 $\mathbf{g}_0 = \mathbf{0}$, $\mathbf{g}_1 = \mathbf{0}$;
4 $\mathbf{y}_T = \mathbf{0}$, $k = 20$;
5 $\nu = 5\times 10^{-2}$;
6 $\mathbf{y}_0$ corresponds to a plane *Poiseuille flow* of maximum velocity equal to 1, i.e.

$$\mathbf{y}_0(x) = \{4x_2(1-x_2), 0\},\ \forall x \in \Omega.$$

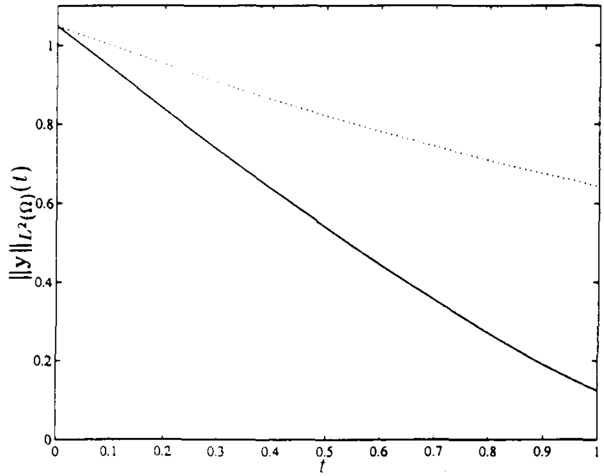


Fig. 36. Variation of $\|\mathbf{y}_h^{\Delta t}(t)\|_{(L^2(\Omega))^2}$ with (—) and without (...) control.

Integrating equations (3.60)–(3.63) with $\mathbf{v} = \mathbf{0}$ will lead to a solution that decays in time with a rate determined by the size of the viscosity parameter ν. The problem here is to find – via (3.59) – a control that will speed up this decay as much as possible at time T.

The *time discretization* has been obtained through a variant of scheme (3.56)–(3.58), using $\Delta t = 1/50$; the space discretization was achieved using a *finite element* approximation associated with a 32×16 (respectively (16×8)) regular grid for the *velocity* (respectively the pressure) (see Berggren and Glowinski (1994), for details). A *fully discrete* variant of the conjugate gradient algorithm (3.38)–(3.54) was used to compute the *approximate optimal control* $\mathbf{u}_h^{\Delta t}$ and the associated *velocity field* $\mathbf{y}_h^{\Delta t}$.

On Figure 36 we compare the decays between $t = 0$ and $t = T = 1$ of the noncontrolled flow velocity (...) and of the controlled flow velocity (—) (we have shown the values of $(\int_\Omega |\mathbf{y}(t)|^2 \, dx)^{1/2}$; remember that $\frac{1}{2} \int_\Omega |\mathbf{y}(t)|^2 \, dx$ is the *flow kinetic* energy). On Figure 37, we have compared, at time T, the *kinetic energy* distributions of the controlled flow (lower graph) and of the noncontrolled one (upper graph). Control has been effective to reduce the flow kinetic energy, particularly on the support $\mathcal{O}$ of the optimal control (according to Figure 37, at least). The results displayed on the following figures were obtained after 70 iterations.

Finally, we have shown on Figure 38 the graph of the first component of the computed optimal control $\mathbf{u}_h^{\Delta t}$ at various values of t.

For further details and comments about these computations see Berggren and Glowinski (1994), where further numerical experiments are also discussed.

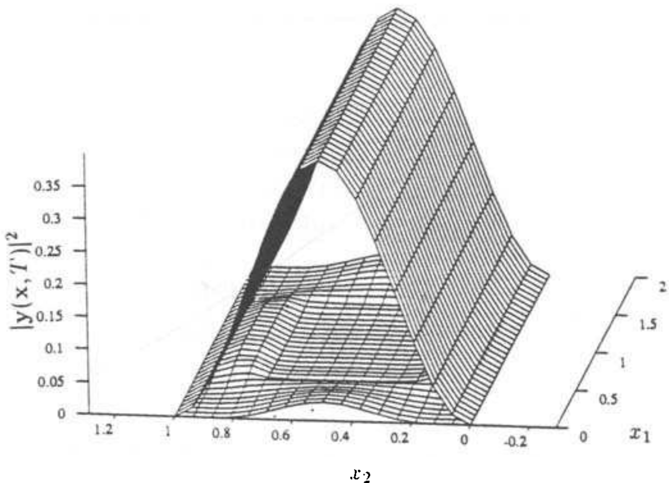


Fig. 37. Kinetic energy distribution of the controlled flow (lower graph) and noncontrolled flow (upper graph). Kinetic energy distribution of the controlled flow (lower graph) and noncontrolled flow (upper graph).

4. CONTROL OF NONLINEAR DIFFUSION SYSTEMS

4.1. Generalities. Synopsis

The various *controllability problems* which have been discussed so far have all been associated with systems governed by linear diffusion equations.

In this section we briefly address the *nonlinear* situation and would like to show that *nonlinearity* may bring *noncontrollability* (as seen in Section 4.2) and also to discuss (in Section 4.3) the solution of *pointwise control problems* for the *viscous Burgers equation.*

Further information is given in V. Komornik (1994), J.L. Lions (1991a), I. Lasiecka (1992), I. Lasiecka and R. Tataru (1994), E. Zuazua (1988) and the references therein.

4.2. An example of a noncontrollable nonlinear system

In this section, we want to emphasize that *approximate controllability is very unstable under 'small' nonlinear perturbations.*

Let us consider again the state equation

$$\frac{\partial y}{\partial t} - \Delta y = v\chi_{\mathcal{O}} \text{ in } Q, \quad y(0) = 0, \; y = 0 \text{ on } \Sigma, \tag{4.1}$$

which is the same equation as in Section 1.1, but where we take $A = -\Delta$ to make things as simple as possible.

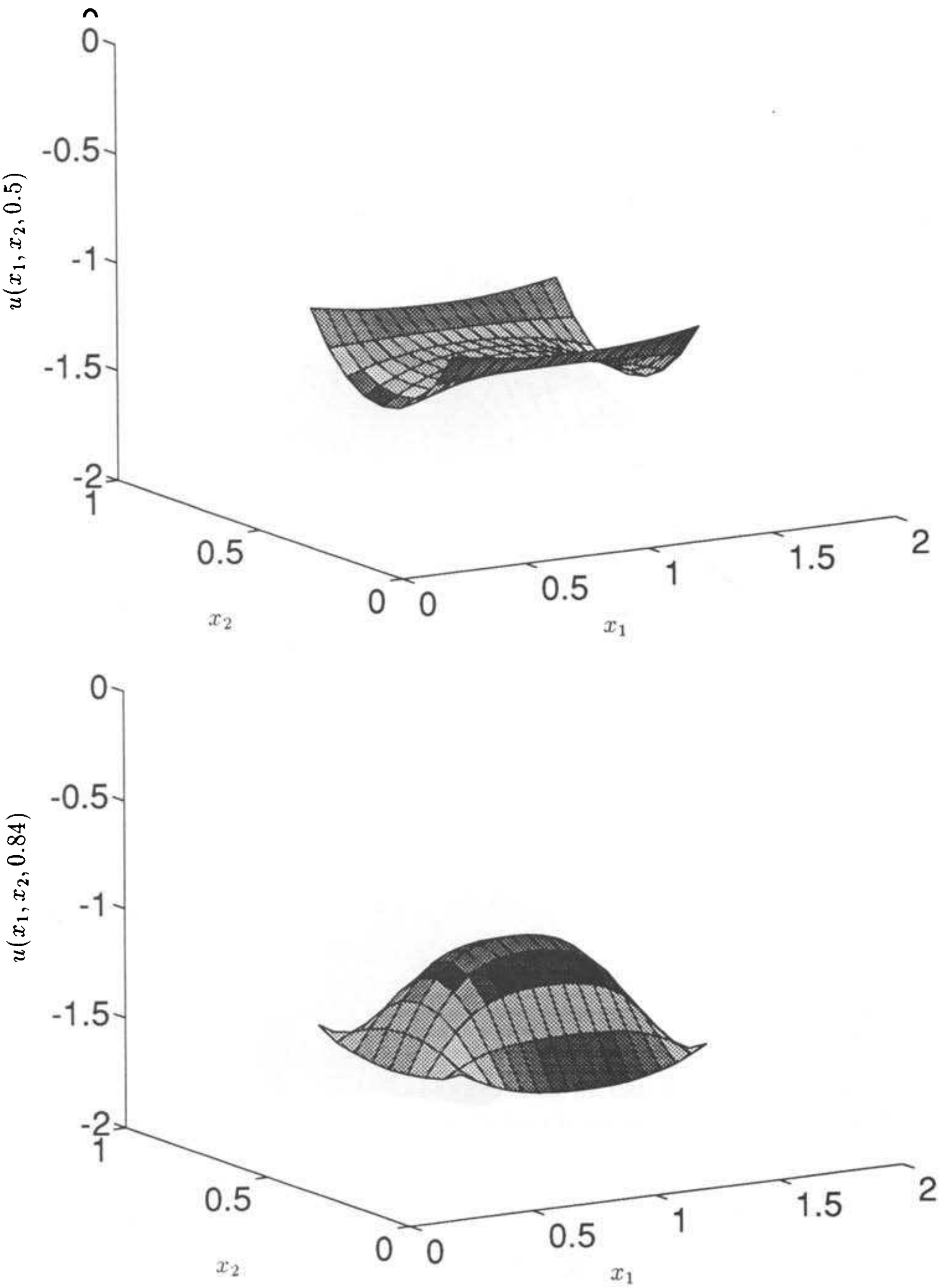


Fig. 38. (a) Graph of the computed optimal control ($t = 0.5$). (b) Graph of the computed optimal control ($t = 0.84$).

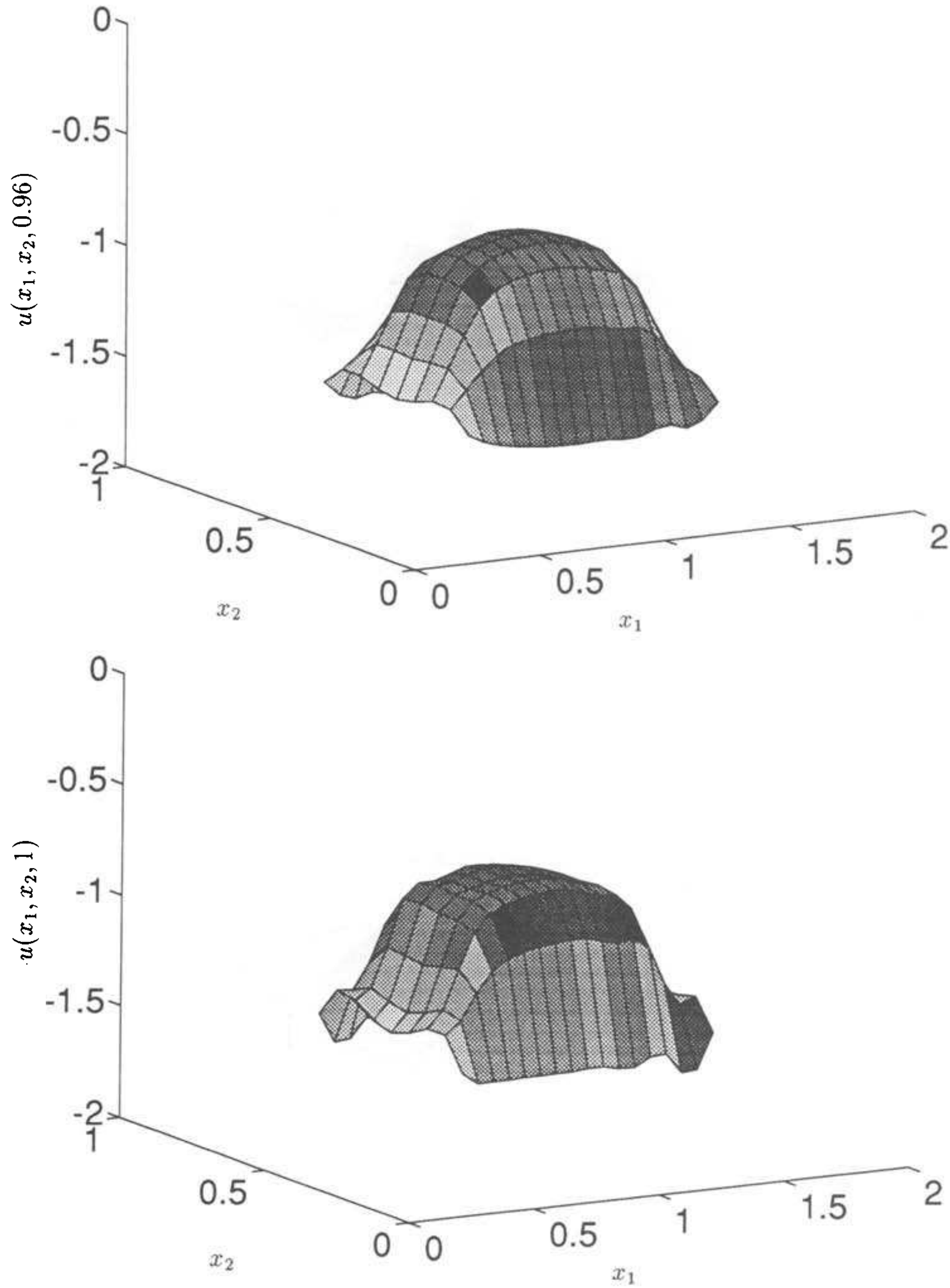


Fig. 38 (*cont.*) (c) Graph of the computed optimal control ($t = 0.96$). (d) Graph of the computed optimal control ($t = 1.0$).

We consider now the *nonlinear* partial differential equation

$$\frac{\partial y}{\partial t} - \Delta y + \alpha y^3 = v\chi_{\mathcal{O}} \text{ in } Q, \quad y(0) = 0, \ y = 0 \text{ on } \Sigma, \tag{4.2}$$

where α is *positive*, otherwise arbitrarily small. Problem (4.2) has a *unique* solution (see, e.g., Lions (1969)). Contrary to what happens for (4.1), *the set described by* $y(T;v)$ ($y(v)$ is the solution of (4.2)) when v spans $L^2(\mathcal{O}\times(0,T))$ *is far from being dense in* $L^2(\Omega)$.

There are several proofs of this result, some of them based on *maximum principles.* The following one is due to A. Bamberger (1977) and is reported in the PhD thesis of Henry (1978). It is based on a simple *energy estimate.* One multiplies (4.2) by my, where $m(x) \geq 0$, $m \equiv 0$ near $\mathcal{O}$, $m \in C^1(\bar{\Omega})$. Then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega my^2 \,\mathrm{d}x + \int_\Omega m|\nabla y|^2 \,\mathrm{d}x + \int_\Omega y\nabla y\cdot\nabla m \,\mathrm{d}x + \alpha\int_\Omega my^4 \,\mathrm{d}x = 0. \tag{4.3}$$

Let us write

$$\int_\Omega y\nabla y\cdot\nabla m \,\mathrm{d}x = \int_\Omega m^{1/4}y(m^{1/2}\nabla y)\cdot(m^{-3/4}\nabla m)\,\mathrm{d}x$$

so that there exists a constant C such that

$$\left|\int_\Omega y\nabla y\cdot\nabla m \,\mathrm{d}x\right| \leq \alpha\int_\Omega my^4 \,\mathrm{d}x + \int_\Omega m|\nabla y|^2 \,\mathrm{d}x + C\int_\Omega m^{-3}|\nabla m|^4 \,\mathrm{d}x. \tag{4.4}$$

Combining (4.3) and (4.4) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega my^2 \,\mathrm{d}x \leq C\int_\Omega m^{-3}|\nabla m|^4 \,\mathrm{d}x$$

so that

$$\frac{1}{2}\int_\Omega m(x)|y(x,T;v)|^2 \,\mathrm{d}x \leq CT\int_\Omega m^{-3}|\nabla m|^4 \,\mathrm{d}x \tag{4.5}$$

no matter how v is chosen, since the right-hand side of (4.5) *does not depend on v.* Of course, this calculation assumes that we can choose m as above and such that $\int_\Omega m^{-3}|\nabla m|^4 \,\mathrm{d}x < +\infty$; such functions m are easy to construct.

Remark 4.1 Examples and counter examples of controllability for nonlinear diffusion type equations are given in Diaz (1991).

4.3. Pointwise control of the viscous Burgers equation

4.3.1. Motivation

The *inviscid* or *viscous Burgers equations* have, for many years, attracted the attention of many investigators, from both the theoretical and numerical points of view. There are several reasons for this 'popularity', one of them being certainly that the Burgers equations provide not too unrealistic simplifications of the *Euler* and *Navier–Stokes* equations of *Fluid Dynamics*; among the features, common with those more complicated equations, *nonlinearity* is certainly the most important single one. It is not surprising, therefore, that the Burgers equations have also attracted the attention of the *Control Community* (see, e.g., Burns and Kang (1991), Burns and Marrekchi (1993)). The present section is another contribution in that direction: we shall address here the solution of controllability problems for the *viscous Burgers equation* via *pointwise controls*; from that point of view this section can be seen as a generalization of Section 1.10 where we addressed the pointwise control of linear diffusion systems (the viscous Burgers equation considered here belongs to the class of the *nonlinear advection–diffusion* systems whose most celebrated representative is the *Navier–Stokes equation* system).

4.3.2. Formulation of the control problems

As in Berggren and Glowinski (1994) (see also Berggren (1992) and Dean and Gubernatis (1991)) we can consider the following *pointwise control problem* for the *viscous Burgers equation*

$$\min_{\mathbf{v}\in\mathcal{U}}\left[\frac{1}{2}\|\mathbf{v}\|_{\mathcal{U}}^2+\frac{1}{2}k\|y(T)-y_T\|_{L^2(0,1)}^2\right], \tag{4.6}$$

where, in (4.6), we have:

1 $\mathbf{v}=\{v_m\}_{m=1}^M$, $\mathcal{U}=L^2(0,T;\mathbb{R}^M)$, $\|\mathbf{v}\|_{\mathcal{U}}=(\sum_{m=1}^M\int_0^T|v_m|^2\,\mathrm{d}t)^{1/2}$;
2 $k>0$, arbitrarily large;
3 $y_T\in L^2(0,1)$ and $y(T)$ is obtained from $\mathbf{v}$ via the solution of the *viscous Burgers equation*, below

$$\frac{\partial y}{\partial t}-\nu\frac{\partial^2 y}{\partial x^2}+y\frac{\partial y}{\partial x}=f+\sum_{m=1}^M v_m\delta(x-a_m)\text{ in }Q(=(0,1)\times(0,T)), \tag{4.7}$$

$$\frac{\partial y}{\partial x}(0,t)=0, y(1,t)=0\ \text{ a.e. on }(0,T), \tag{4.8}$$

$$y(0)=y_0(\in L^2(0,1)); \tag{4.9}$$

in (4.7), $\nu(>0)$ is a *viscosity parameter*, f a *forcing term*, $a_m\in(0,1)$, $\forall m=1,\ldots,M$ and $x\to\delta(x-a_m)$ denotes the *Dirac* measure at a_m.

Let us denote by V the (Sobolev) space defined by

$$V=\{z\mid z\in H^1(0,1), z(1)=0\}, \tag{4.10}$$

and suppose that $f\in L^2(0,T;V')$ (V': dual space of V); it follows then from Lions (1969) that for $\mathbf{v}$ given in $\mathcal{U}$ the Burgers system (4.7)–(4.9) has a *unique solution* in $L^2(0,T;V)\cap C^0([0,T];L^2(0,1))$. From this result, we can show that the control problem (4.6) *has a solution* (not *necessarily unique*, due to the *nonconvexity* of the functional $J:\mathcal{U}\to\mathbb{R}$, where J is the *cost function* in (4.6)).

Remark 4.2 In Glowinski and Berggren (1994), we have discussed the solution of the variant of problem (4.6) where the location on (0,1) of the a_m's is *unknown* (a_m: 'support' of the mth pointwise control). The solution methods described in the following can easily be modified to accommodate this more complicated situation (see the above reference for details and numerical results).

4.3.3. Optimality conditions for problem (4.6)

To compute a control $\mathbf{u}$ solution of problem (4.6) we shall derive first *necessary optimality conditions* and use them (in Section 4.3.4) through a *conjugate gradient algorithm* to obtain the above solution.

The derivative $J'(\mathbf{v})$ of J at $\mathbf{v}$ can be obtained from

$$(J'(\mathbf{v}), \mathbf{w})_{\mathcal{U}} = \lim_{\substack{\theta \to 0 \\ \theta \neq 0}} \frac{J(\mathbf{v} + \theta \mathbf{w}) - J(\mathbf{v})}{\theta}. \tag{4.11}$$

Actually, instead of (4.11), we shall use a (formal) perturbation analysis to obtain $J'(\mathbf{v})$:

First, we have

$$\delta J(\mathbf{v}) = (J'(\mathbf{v}), \delta \mathbf{v})_{\mathcal{U}} = \sum_{m=1}^{M} \int_0^T v_m \delta v_m \, \mathrm{d}t + k \int_0^1 (y(T) - y_T) \delta y(T) \, \mathrm{d}x \tag{4.12}$$

where (from (4.7)–(4.9)) $\delta y(T)$ is obtained from $\delta \mathbf{v}$ via the solution of the following variational problem

$$\delta y(t) \in V \text{ a.e. on } (0,T); \ \forall z \in V \text{ we have a.e. on } (0,T)$$

$$\left\langle \frac{\partial}{\partial t} \delta y, z \right\rangle + \nu \int_0^1 \frac{\partial}{\partial x} \delta y \frac{\partial z}{\partial x} \, \mathrm{d}x + \int_0^1 \delta y \frac{\partial y}{\partial x} z \, \mathrm{d}x + \int_0^1 y \frac{\partial}{\partial x} \delta y z \, \mathrm{d}x = \sum_{m=1}^{m} \delta v_m z(a_m), \tag{4.13}$$

$$\delta y(0) = 0; \tag{4.14}$$

in (4.13), $\langle \cdot, \cdot \rangle$ denotes the *duality pairing* between V' and V.

Consider now $p \in L^2(0,T;V) \cap C^0([0,T]; L^2(0,1))$ such that $\partial p / \partial t \in L^2(0,T;V')$; taking $z \in p(t)$ in (4.13) we obtain

$$\int_0^T \left\langle \frac{\partial}{\partial t} \delta y, p \right\rangle \mathrm{d}t + \nu \int_0^T \int_0^1 \frac{\partial}{\partial x} \delta y \frac{\partial p}{\partial x} \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_0^1 \left(\delta y \frac{\partial y}{\partial x} + y \frac{\partial}{\partial x} \delta y \right) p \, \mathrm{d}x \, \mathrm{d}t = \sum_{m=1}^{M} \int_0^T p(a_m, t) \delta v_m \, \mathrm{d}t. \tag{4.15}$$

Integrating by parts over $(0,T)$ it follows from (4.14), (4.15) that

$$\int_0^1 p(T)\delta y(T)\,\mathrm{d}x - \int_0^T \left\langle \frac{\partial p}{\partial t}, \delta y \right\rangle \mathrm{d}t + \nu \int_0^T \int_0^1 \frac{\partial}{\partial x}\delta y \frac{\partial p}{\partial x}\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^T \int_0^1 \left(\delta y \frac{\partial y}{\partial x} + y \frac{\partial}{\partial x}\delta y \right) p\,\mathrm{d}x\,\mathrm{d}t$$
$$= \sum_{m=1}^{M} \int_0^T p(a_m, t)\delta v_m\,\mathrm{d}t. \tag{4.16}$$

Suppose now that p satisfies also

$$-\left\langle \frac{\partial p}{\partial t}, z \right\rangle + \nu \int_0^1 \frac{\partial p}{\partial x}\frac{\partial z}{\partial x}\,\mathrm{d}x + \int_0^1 p \left(\frac{\partial y}{\partial x} z + y \frac{\partial z}{\partial x} \right) \mathrm{d}x = 0,$$
$$\forall z \in V, \text{ a.e. on } (0,T), \tag{4.17}$$

and

$$p(T) = k(y_T - y(T)); \tag{4.18}$$

it follows then from (4.16) that

$$k \int_0^1 (y(T) - y_T)\delta y(T)\,\mathrm{d}x = -\sum_{m=1}^{M} \int_0^T p(a_m, t)\delta v_m\,\mathrm{d}t,$$

which combined with (4.12) implies in turn that

$$(J'(\mathbf{v}), \delta \mathbf{v})_{\mathcal{U}} = \sum_{m=1}^{M} \int_0^T (v_m(t) - p(a_m, t))\delta v_m(t)\,\mathrm{d}t.$$

We have thus 'proved' that, $\forall \mathbf{v},\ \mathbf{w} \in \mathcal{U}$

$$(J'(\mathbf{v}), \mathbf{w})_{\mathcal{U}} = \sum_{m=1}^{M} \int_0^T (v_m(t) - p(a_m, t))w_m(t)\,\mathrm{d}t. \tag{4.19}$$

Remark 4.3 Starting from (4.11) we can give a rigorous proof of (4.19).

Suppose now that $\mathbf{u}$ is a solution of problem (4.6); we have then $J'(\mathbf{u}) = \mathbf{0}$ which provides the following *optimality system*

$$u_m(t) = p(a_m, t),\ \forall m = 1, \dots, M, \text{ on } (0,T), \tag{4.20}$$

$$\frac{\partial y}{\partial t} - \nu \frac{\partial^2 y}{\partial x^2} + y \frac{\partial y}{\partial x} = f + \sum_{m=1}^{M} u_m \delta(x - a_m) \text{ in } Q, \tag{4.21}$$

$$\frac{\partial y}{\partial x}(0,t) = 0,\ y(1,t) = 0 \text{ on } (0,T), \tag{4.22}$$

$$y(0) = y_0 \tag{4.23}$$

$$-\frac{\partial p}{\partial t} - \nu \frac{\partial^2 p}{\partial x^2} - y\frac{\partial p}{\partial x} = 0 \text{ in } Q, \tag{4.24}$$

$$\nu \frac{\partial p}{\partial x}(0,t) + y(0,t)p(0,t) = 0, \ p(1,t) = 0 \text{ on } (0,T), \tag{4.25}$$

$$p(T) = k(y_T - y(T)). \tag{4.26}$$

4.3.4. Iterative solution of the control problem (4.6)

Conjugate gradient algorithms are particularly attractive for large scale nonlinear problems since their applications requires only – in principle – *first derivative information* (see, e.g., Daniel (1970), Polack (1971) and Nocedal (1992) for further comments and convergence proofs). Problem (4.6) is a particular case of the minimization problem

$$\begin{cases} u \in H, \\ j(u) \le j(v), \ \forall v \in H, \end{cases} \tag{4.27}$$

where H is a real Hilbert space for the scalar product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$ and where the functional $j : H \to \mathbb{R}$ is *differentiable*; we denote by $j'(v)$ ($\in H'; H'$: dual space of H) the differential of j at v.

A *conjugate gradient* algorithm for solving (4.27) is defined as follows:

$$u^0 \text{ is given in } H; \tag{4.28}$$

solve

$$\begin{cases} g^0 \in H, \\ (g^0, v) = \langle j'(u^0), v\rangle, \ \forall v \in H, \end{cases} \tag{4.29}$$

and set

$$w^0 = g^0. \tag{4.30}$$

For $n \ge 0$, assuming that u^n, g^n, w^n are known, compute u^{n+1}, g^{n+1}, w^{n+1} by

$$\begin{cases} \textit{Find } \rho_n \in \mathbb{R} \ \textit{ such that} \\ j(u^n - \rho_n w^n) \le j(u^n - \rho w^n), \ \forall \rho \in \mathbb{R}, \end{cases} \tag{4.31}$$

set

$$u^{n+1} = u^n - \rho_n w^n, \tag{4.32}$$

and solve

$$\begin{cases} g^{n+1} \in H, \\ (g^{n+1}, v) = \langle j'(u^{n+1}), v\rangle, \ \forall v \in H. \end{cases} \tag{4.33}$$

If $\|g^{n+1}\|/\|g^0\| \le \epsilon$ take $u = u^{n+1}$; else compute either

$$\gamma_n = \frac{\|g^{n+1}\|^2}{\|g^n\|^2} \ \textit{(Fletcher–Reeves update)} \tag{4.34$_1$}$$

or

$$\gamma_n = \frac{(g^{n+1}, g^{n+1} - g^n)}{\|g^n\|^2} \quad \textit{(Polack–Ribière update)} \tag{4.34$_2$}$$

and then

$$w^{n+1} = g^{n+1} + \gamma_n w^n. \tag{4.35}$$

Do $n = n + 1$ *and go to* (4.31).

We observe that each iteration requires the solution of a *linear problem* ((4.29) for $n = 0$, (4.33) for $n \geq 1$) and the *line search* (4.31). In most applications the Polack–Ribière variant of algorithm (4.28)–(4.35) is faster than the Fletcher–Reeves one (see, e.g,, Powell (1976) for an explanation of this fact).

Application to problem (4.6). Problem (4.6) is a particular case of (4.27) where $H = \mathcal{U} = L^2(0,T;\mathbb{R}^M)$; combining (4.19) and (4.28)–(4.35) we obtain the following solution method for problem (4.6):

$$\mathbf{u}^0 \text{ is given in } \mathcal{U}; \tag{4.36}$$

solve

$$\begin{cases} \dfrac{\partial y^0}{\partial t} - \nu \dfrac{\partial^2 y^0}{\partial x^2} + y^0 \dfrac{\partial y^0}{\partial x} = f + \sum\limits_{m=1}^{M} u_m^0 \delta(x - a_m) \text{ in } Q, \\ \dfrac{\partial y^0}{\partial x}(0,t) = 0, y^0(1,t) = 0 \text{ on } (0,T),\ y^0(0) = y_0, \end{cases} \tag{4.37}$$

and

$$\begin{cases} -\dfrac{\partial p^0}{\partial t} - \nu \dfrac{\partial^2 p^0}{\partial x^2} - y^0 \dfrac{\partial p^0}{\partial x} = 0 \text{ in } Q, \\ \nu \dfrac{\partial p^0}{\partial x}(0,t) + y^0(0,t) p^0(0,t) = 0,\ p^0(1,t) = 0 \text{ on } (0,T), \end{cases} \tag{4.38$_1$}$$

$$p^0(T) = k(y_T - y^0(T)). \tag{4.38$_2$}$$

Solve then

$$\begin{cases} \mathbf{g}^0 \in \mathcal{U};\ \forall \mathbf{v} \in \mathcal{U}, \text{ we have} \\ \displaystyle\int_0^T \mathbf{g}^0 \cdot \mathbf{v}\, dt = \sum_{m=1}^{M} \int_0^T (u_m^0(t) - p^0(a_m,t)) v_m(t)\, dt, \end{cases} \tag{4.39}$$

and set

$$\mathbf{w}^0 = \mathbf{g}^0. \tag{4.40}$$

Then for $n \geq 0$, *assuming that* $\mathbf{u}^n$, $\mathbf{g}^n$, $\mathbf{w}^n$ *are known compute* $\mathbf{u}^{n+1}$, $\mathbf{g}^{n+1}$, $\mathbf{w}^{n+1}$ *as follows.*

Solve the following one-dimensional minimization problem

$$\begin{cases} \rho_n \in \mathbb{R}, \\ J(\mathbf{u}^n - \rho_n \mathbf{w}^n) \le J(\mathbf{u}^n - \rho \mathbf{w}^n), \ \forall \rho \in \mathbb{R}, \end{cases} \tag{4.41}$$

and update $\mathbf{u}^n$ *by*

$$\mathbf{u}^{n+1} = \mathbf{u}^n - \rho_n \mathbf{w}^n. \tag{4.42}$$

Next, solve

$$\begin{cases} \dfrac{\partial y^{n+1}}{\partial t} - \nu \dfrac{\partial^2 y^{n+1}}{\partial x^2} = f + \sum\limits_{m=1}^{M} u_m^{n+1} \delta(x - a_m) \text{ in } Q, \\ \dfrac{\partial y^{n+1}}{\partial x}(0,t) = 0, y^{n+1}(1,t) = 0 \text{ on } (0,T), \ y^{n+1}(0) = y_0, \end{cases} \tag{4.43}$$

and

$$\begin{cases} -\dfrac{\partial p^{n+1}}{\partial t} - \nu \dfrac{\partial^2 p^{n+1}}{\partial x^2} - y^{n+1} \dfrac{\partial p^{n+1}}{\partial x} = 0 \text{ in } Q, \\ \nu \dfrac{\partial p^{n+1}}{\partial x}(0,t) + y^{n+1}(0,t) p^{n+1}(0,t) = 0, \ p^{n+1}(1,t) = 0 \text{ on } (0,T), \end{cases} \tag{4.44$_1$}$$

$$p^{n+1}(T) = k(y_T - y^{n+1}(T)). \tag{4.44$_2$}$$

Solve then

$$\begin{cases} \mathbf{g}^{n+1} \in \mathcal{U}; \ \forall \mathbf{v} \in \mathcal{U}, \quad \textit{we have} \\ \displaystyle\int_0^T \mathbf{g}^{n+1} \cdot \mathbf{v} \, dt = \sum_{m=1}^{M} \int_0^T (u_m^{n+1}(t) - p^{n+1}(a_m, t)) v_m(t) \, dt. \end{cases} \tag{4.45}$$

If $\|\mathbf{g}^{n+1}\|_{\mathcal{U}} / \|\mathbf{g}^0\|_{\mathcal{U}} \le \epsilon$ *take* $\mathbf{u} = \mathbf{u}^{n+1}$*; else compute either*

$$\gamma_n = \int_0^T |\mathbf{g}^{n+1}|^2 \, dt \Big/ \int_0^T |\mathbf{g}^n|^2 \, dt \ \textit{(Fletcher–Reeves)} \tag{4.46$_1$}$$

or

$$\gamma_n = \int_0^T \mathbf{g}^{n+1} \cdot (\mathbf{g}^{n+1} - \mathbf{g}^n) \, dt \Big/ \int_0^T |\mathbf{g}^n|^2 \, dt \ \textit{(Polack–Ribière)} \tag{4.46$_2$}$$

and update $\mathbf{w}^n$ *by*

$$\mathbf{w}^{n+1} = \mathbf{g}^{n+1} + \gamma_n \mathbf{w}^n. \tag{4.47}$$

Do $n = n + 1$ *and go to* (4.41).

The practical implementation of algorithm (4.36)–(4.47) will rely on the *numerical integration* of the parabolic problems (4.37), (4.38), (4.43), (4.44) (to be discussed in Section 4.3.5) and on the efficiency and accuracy of the *line search* (4.41); actually, to solve the nonlinear problem (4.41) we have employed the *cubic backtracking* strategy advocated in Dennis and Schnabel (1983, Ch. 6).

4.3.5. Space–time discretization of the control problem (4.6). Optimality conditions

We shall use a combination of *finite element* and *finite difference* methods for the space-time discretization of problem (4.6); for simplicity, we shall consider *uniform meshes* for both discretizations. We consider therefore two positive integers I and Δt (to be 'large' in practice) and define the discretization steps h and Δt by $h = 1/I$, $\Delta t = T/N$. Next, we define $x_i = ih$, $i = 0, 1, \ldots, I$ and approximate $L^2(0,1)$ and $H^1(0,1)$ by

$$H_h^1 = \{z_h \mid z_h \in C^0[0,1], z_h|_{[x_{i-1},x_i]} \in P_1, \ \forall i = 1, \ldots, I\},$$

where P_1 denotes the space of the polynomials in one variable of degree less than or equal to one. The space V in (4.10) is approximated by

$$V_h = \{z_h \mid z_h \in H_h^1, z_h(1) = 0\} \ (= V \cap H_h^1),$$

while the *control space* $\mathcal{U}(= L^2(0,T;\mathbb{R}^M))$ in (4.6) is approximated by

$$\mathcal{U}^{\Delta t} = (\mathbb{R}^M)^N = \left\{\mathbf{v} \mid \mathbf{v} = \{\{v_m^n\}_{m=1}^M\}_{n=1}^N\right\}, \tag{4.48}$$

to be equipped with the following scalar product

$$(\mathbf{v}, \mathbf{w})_{\Delta t} = \Delta t \sum_{n=1}^{N} \sum_{m=1}^{M} v_m^n w_m^n, \ \forall \mathbf{v}, \mathbf{w} \in \mathcal{U}^{\Delta t}.$$

We approximate then the control problem (4.6) by

$$\min_{\mathbf{v} \in \mathcal{U}^{\Delta t}} J_h^{\Delta t}(\mathbf{v}), \tag{4.49}$$

where the functional $J_h^{\Delta t} : \mathcal{U}^{\Delta t} \to \mathbb{R}$ is defined by

$$J_h^{\Delta t}(\mathbf{v}) = \tfrac{1}{2}(\mathbf{v}, \mathbf{v})_{\Delta t} + \tfrac{1}{2}k\|y^N - y_T\|_{L^2(0,1)}^2, \tag{4.50}$$

with y^N defined from $\mathbf{v}$ via the solution of the following *discrete Burgers equation*:

$$y^0 = y_{0h} \in H_h^1 \ \text{ such that } \lim_{h \to 0} \|y_{0h} - y_0\|_{L^2(0,1)} = 0; \tag{4.51}$$

for $n = 1, \ldots, N$ we obtain y^n from y^{n-1} via the solution of the following discrete linear (elliptic) variational problem

$$\begin{cases} y^n \in V_h; \ \forall z \in V_h \ \text{ we have} \\ \displaystyle\int_0^1 \frac{y^n - y^{n-1}}{\Delta t} z \, dx + \nu \int_0^1 \frac{dy^n}{dx}\frac{dz}{dx} \, dx + \int_0^1 y^{n-1}\frac{dy^{n-1}}{dx} z \, dx \\ \displaystyle = \int_0^1 f z \, dx + \sum_{m=1}^{M} v_m^n z(a_m). \end{cases} \tag{4.52}$$

Scheme (4.51), (4.52) is *semi-implicit* since the nonlinear term $y(\mathrm{d}y/\mathrm{d}x)$ is treated *explicitly*; we can expect therefore that Δt has to satisfy a *stability* condition. It is easily verified that obtaining y^n from y^{n-1} is *equivalent* to solving a *linear system* for a matrix (the discrete analogue of operator $(I/\Delta t) - \nu \mathrm{d}^2/\mathrm{d}x^2$) which is *tridiagonal, symmetric* and *positive definite.* If Δt is constant over the time interval $(0, T)$ this matrix being independent of n can be *Cholesky* factored once for all.

The *approximate control problem* (4.49) *has at least one solution* $\mathbf{u}_h^{\Delta t} = \{\{u_m^n\}_{m=1}^M\}_{n=1}^N$. Any solution of problem (4.49) satisfies the (necessary) *optimality condition*

$$\nabla J_h^{\Delta t}(\mathbf{u}_h^{\Delta t}) = 0, \tag{4.53}$$

where $\nabla J_h^{\Delta t}$ is the *gradient* of the functional $J_h^{\Delta t}$.

Following the approach taken in Section 4.3.3 for the continuous problem (4.6) we can show that

$$(\nabla J_h^{\Delta t}(\mathbf{v}), \mathbf{w}) = \Delta t \sum_{n=1}^{N} \sum_{m=1}^{M} (v_m^n - p^n(a_m)) w_m^n, \ \forall \mathbf{v}, \mathbf{w} \in \mathcal{U}^{\Delta t}, \tag{4.54}$$

where $\{p^n\}_{n=1}^N$ is obtained from $\mathbf{v}$ via the solution of the discrete Burgers equation (4.51), (4.52), followed by the solution of the *discrete adjoint* equation, below.

Compute

$$p^{N+1} \in V_h \ \text{ such that } \int_0^1 p^{N+1} z \,\mathrm{d}x = k \int_0^1 (y_T - y^N) z \,\mathrm{d}x, \ \forall z \in V_h, \tag{4.55}$$

and then for $n = N, \ N-1, \dots, 1, \ p^n$ is obtained from p^{n+1} via the solution of the discrete elliptic problem

$$\left\{ \begin{array}{l} p^n \in V_h; \ \text{ we have} \\ \displaystyle\int_0^1 \frac{p^n - p^{n+1}}{\Delta t} z \,\mathrm{d}x + \nu \int_0^1 \frac{\mathrm{d}p^n}{\mathrm{d}x} \frac{\mathrm{d}z}{\mathrm{d}x} \,\mathrm{d}x + \int_0^1 p^{n+1} \left(y^n \frac{\mathrm{d}z}{\mathrm{d}x} + \frac{\mathrm{d}y^n}{\mathrm{d}x} z \right) \mathrm{d}x = 0. \end{array} \right. \tag{4.56}$$

The comments concerning the calculation of y^n from y^{n-1} still apply here; actually, the linear systems to be solved at each time step to obtain p^n from p^{n+1} have the same matrix as those encountered in the calculation of y^n from y^{n-1}.

From (4.54), we can derive a fully discrete variant of algorithm (4.36)–(4.47) to solve the approximate control problem (4.49) via the optimality conditions (4.53); such an algorithm is discussed in Berggren and Glowinski (1994).

4.3.6. Numerical experiments

Following Berggren and Glowinski (1994) (see also Dean and Gubernatis (1991), Glowinski (1991)) we consider particular cases of problem (4.6) which

Table 4. *Summary of numerical results.*

a	1/5	2/3
Number of iterations	89	47
$\frac{\lVert y_h^{\Delta t}(T) - y_T \rVert_{L^2(0,1)}}{\lVert y_T \rVert_{L^2(0,1)}}$	2×10^{-1}	9×10^{-2}
$\lVert u_h^{\Delta t} \rVert_{L^2(0,T)}$	0.11	0.11

have in common:

$$T = 1,\ \nu = 10^{-2},\ k = 8,\ y_0 = 0,$$

$$f(x,t) = \begin{cases} 1 \text{ if } \{x,t\} \in (0,1/2) \times (0,T), \\ 2(1-x) \text{ if } \{x,t\} \in (1/2,1) \times (0,T), \end{cases}$$

$$y_T(x) = 1 - x^2, \text{ if } x \in (0,1).$$

To discretize the corresponding control problems, we have used the methods described in Section 4.3.5 with $h = 1/128$ and $\Delta t = 1/256$. The discrete control problems (4.49) have been solved by the fully discrete variant of algorithm (4.36)–(4.47) mentioned in Section 4.3.5, using $\mathbf{u}^0 = \mathbf{0}$ as an *initial guess* and $\epsilon = 10^{-5}$ as the *stopping criterium.*

First, several experiments were performed with a *single* control point ($M = 1$) for different values of $a(= a_1)$. In Table 4 we have summarized some of the numerical results concerning the *computed optimal control* $u_h^{\Delta t}$ and the corresponding *discrete state function* $y_h^{\Delta t}$:

For $a = 1/5$ (respectively $a = 2/3$) we have visualized on Figure 39(a) (respectively Figure 40(a)) the *computed optimal control* $u_h^{\Delta t}$ while on Figure 39(a) (respectively Figure 40(b)) we have compared the *target function* y_T (...) with the computed approximation $y_h^{\Delta t}(T)$ (—) of $y(T)$.

For $a = 2/3$ a good fit *downstream* from the control point can be noticed, while the solution seems to be *close to uncontrollable upstream.* The positive sign of the solution implies that the *convection* is directed towards the increasing values of x, which is why it seems reasonable that the system is at least locally controllable in that direction. The only way of controlling the system upstream is through the diffusion term, which is small here ($\nu = 10^{-2}$) compared with the convection term. For the case $a = 1/5$ there are clearly problems with controllability far downstream of the controller (recall that there is a distributed, uncontrolled forcing term, f, which affects the solution).

Figure 41 shows the target and the final state when *two* control points

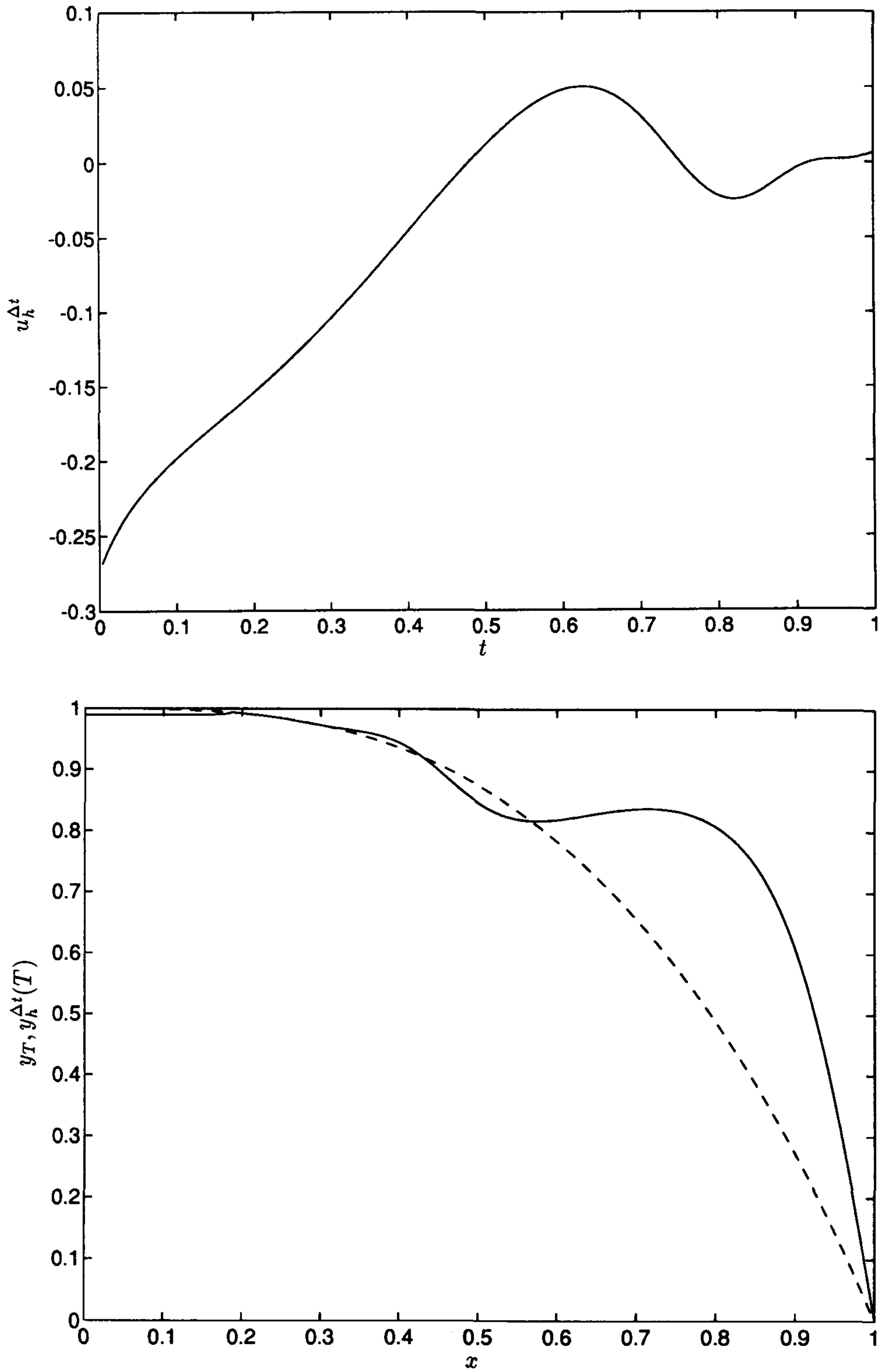


Fig. 39. (a) Graph of the computed control $u_h^{\Delta t}(a = 1/5)$. (b) Comparison between y_T (...) and $y_h^{\Delta t}(T)$ (—) ($a = 1/5$).

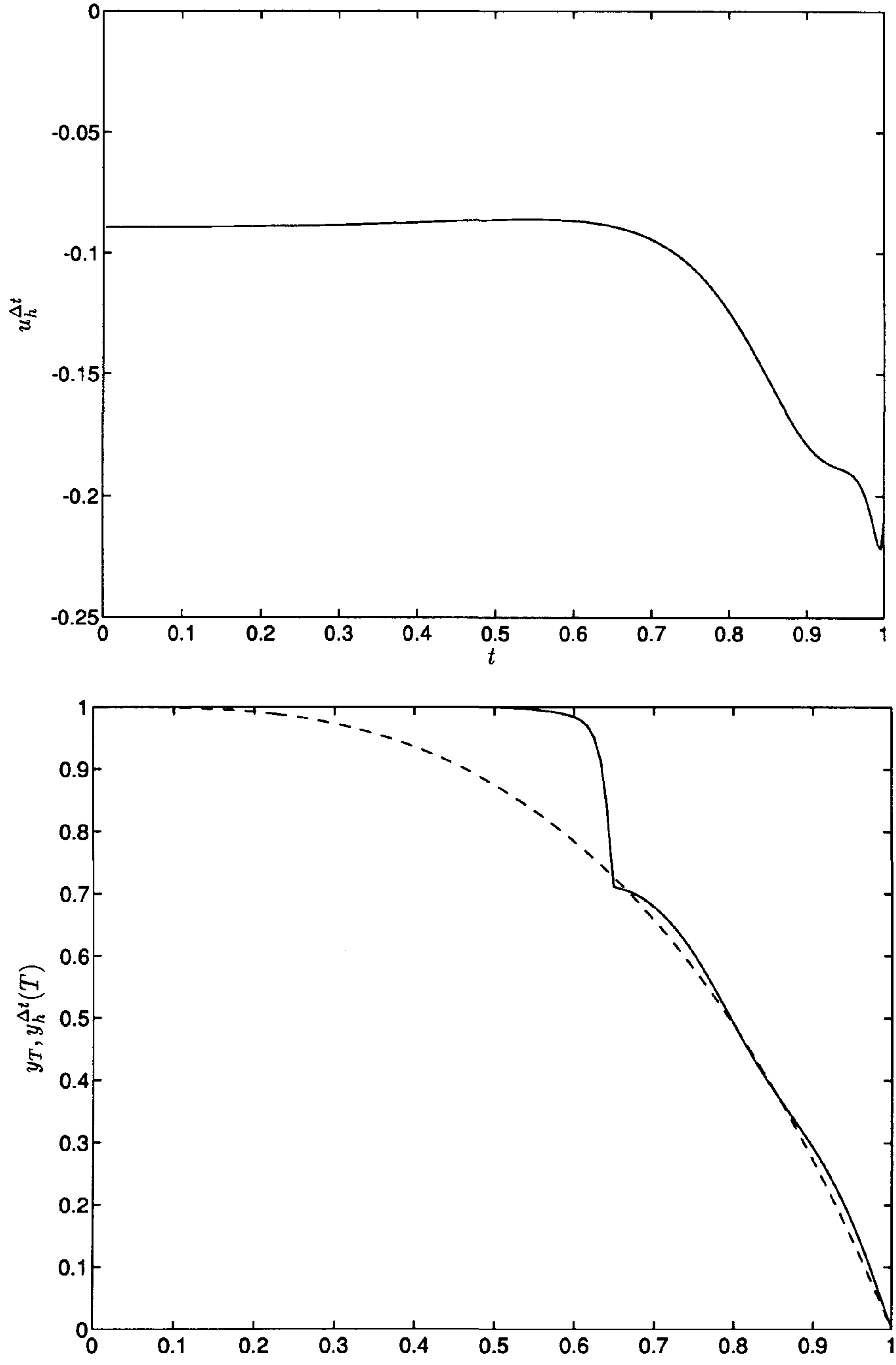


Fig. 40. (a) Graph of the computed control $u_h^{\Delta t}(a = 2/3)$. (b) Comparison between y_T (...) and $y_h^{\Delta t}(T)$ (—) $(a = 2/3)$.

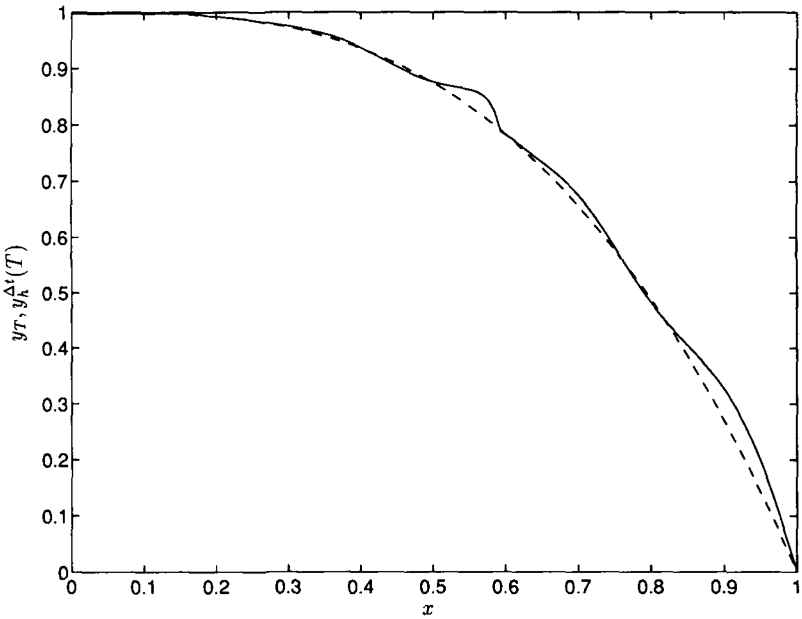


Fig. 41. Comparison between y_T (...) and $y_h^{\Delta t}(T)$ (—) ($\mathbf{a} = \{1/5, 3/5\}$).

Table 5. *Summary of numerical results.*

$\mathbf{a}$	$\{1/5, 3/5\}$	$\{0.1, 0.3, 0.5, 0.7, 0.9\}$
Number of iterations	86	82
$\dfrac{\|y_h^{\Delta t}(T) - y_T\|_{L^2(0,1)}}{\|y_T\|_{L^2(0,1)}}$	2.5×10^{-2}	8.5×10^{-3}

are used, namely $a_1 = 1/5$ and $a_2 = 3/5$; the results are significantly better. Actually the results become 'very good' (as shown on Figure 42) when one uses the *five* control points $a_1 = 0.1$, $a_2 = 0.3$, $a_3 = 0.5$, $a_4 = 0.7$ and $a_5 = 0.9$; in that case we are 'close' to a control distributed over the whole interval (0,1). Some further results are summarized in Table 5.

Remark 4.4 Concerning the convergence of the *conjugate gradient algorithm* used to solve the approximate control problems (4.49) let us mention that

(i) The *Fletcher–Reeves* variant seems to have here a faster convergence than the *Polak–Ribière* one.

(ii) The computational time does not depend too much on the number M of control points. For example the CPU time (user time on a SUN

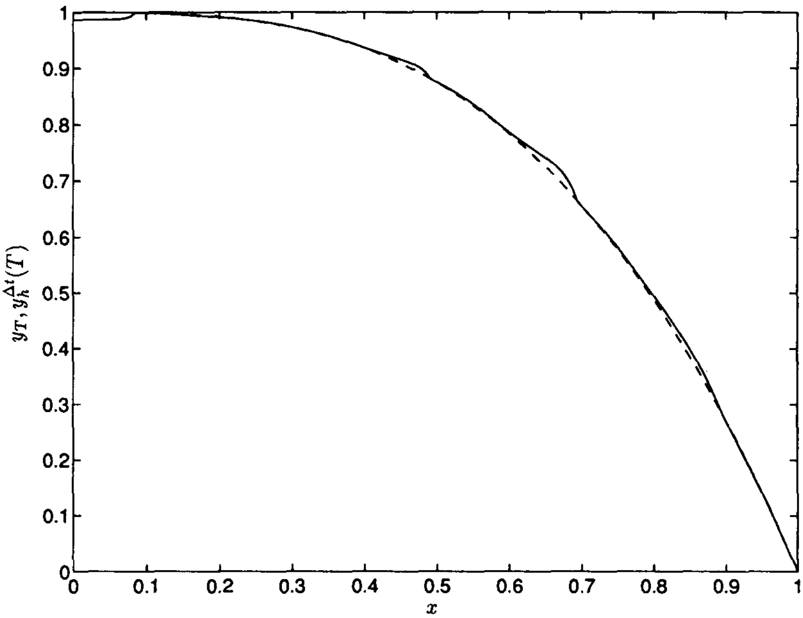


Fig. 42. Comparison between y_T (...) and $y_h^{\Delta t}(T)$ (—) ($\mathbf{a} = \{0.1, 0.3, 0.5, 0.7, 0.9\}$).

Workstation SPARC10) was about 22 s for the case with *one* control point at $a = 1/5$, to be compared with 27 s for the *five* control points test problem. Thus, the time-consuming part is the solution of the discrete state and adjoint state equations and not the manipulation of the control vectors (see Berggren and Glowinski (1994) for further details).

Remark 4.5 In Berggren and Glowinski (1994) we have also addressed and solved the more complicated problem where the control $\mathbf{u}$ *and* the location $\mathbf{a}$ of the controllers are unknown; this new problem can also be solved by a *conjugate gradient algorithm* operating in $L^2(0, T; \mathbb{R}^M) \times \mathbb{R}^M$; compared with the case where $\mathbf{a}$ is fixed the convergence of the new algorithm is much (about 4 times) slower (see, Berggren and Glowinski (1994) for the computational aspects and for numerical results).

4.3.7. Controllability and the Navier–Stokes equations

Flow control is an important part of Engineering and from that point of view has been around for many years. However the corresponding *mathematical problems* are quite difficult and most of them are still open; it is therefore our opinion that a survey on the *numerical aspects* of these problems is still premature.

It is nevertheless worth mentioning that a most important issue in that direction is the *control of turbulence* motivated, for example, by *drag reduc-*

tion (see, e.g., Buschnell and Hefner (1990) and Sellin and Moses (1989)). Another important issue concerns the control of *turbulent combustion* as discussed in, e.g., McManus, Poinsot and Candel (1993) and Samaniego, Yip, Poinsot and Candel (1993).

Despite the lack of theoretical results there is an enormous amount of literature on flow control topics (see, e.g., the four above publications and the references therein). Focusing on recent work in the spirit of the present article, let us mention Abergel and Temam (1990), Lions (1991a), Glowinski (1991), this list being far from complete. In the following we shall give further references; they concern the application of *Dynamic Programming* to the control of system governed by the Navier–Stokes equations.

5. DYNAMIC PROGRAMMING FOR LINEAR DIFFUSION EQUATIONS

5.1. Introduction. Synopsis

We address in this section the '*real time*' aspect of the controllability problems. We proceed in a largely formal fashion. The content of this section is based on Lions (1991b)

We consider again the *state equation*

$$\frac{\partial y}{\partial t} + Ay = v\chi_{\mathcal{O}}, \tag{5.1}$$

now in the time interval $(s, T], 0 \leq s \leq T$; the *'initial' condition* is

$$y(s) = h, \tag{5.2}$$

where h is an *arbitrary* function in $L^2(\Omega)$; the *boundary* condition is

$$y = 0 \text{ on } \Sigma_s = \Gamma \times (s, T). \tag{5.3}$$

Consider now the following *control problem*

$$\inf \frac{1}{2} \iint_{\mathcal{O}\times(s,T)} v^2 \, dx \, dt, \; v \in L^2(\mathcal{O} \times (s,T)) \;\text{ so that } y(T;v) \in y_T + \beta B, \tag{5.4}$$

where in (5.4), $\beta > 0$, B is the closed unit ball of $L^2(\Omega)$ centred at 0, $y_T \in L^2(\Omega)$ and $t \to y(t; v)$ is the solution of (5.1)–(5.3).

The *minimum* in (5.4) is now a function of h and s, we define $\phi(h, s)$ by

$$\phi(h, s) = \text{ minimal value of the cost function in (5.4).} \tag{5.5}$$

We now derive the *Hamilton–Jacobi–Bellman* (HJB) *equation* satisfied by ϕ on $\times(0, T)$.

5.2. Derivation of the Hamilton–Jacobi–Bellman equation

As we said above, we shall proceed in a largely formal fashion. We take

$$v(x,t) = w(x) \text{ in } (s, s+\varepsilon),\ \varepsilon > 0 \text{ `very small'}. \tag{5.6}$$

With this choice of v, the state function $y(t)$ moves during the time interval $(s, s+\varepsilon)$ from h to an element 'very close' to

$$h_\varepsilon = h - \varepsilon A h + \varepsilon w \chi_{\mathcal{O}}, \tag{5.7}$$

assuming that $h \in H^2(\Omega) \cap H_0^1(\Omega)$ (h_ε is obtained from h by the *explicit Euler scheme*).

On the time interval $(s+\varepsilon, T)$ we consider the whole process starting from h_ε at time $s+\varepsilon$. The *optimality principle* leads to

$$\phi(h,s) = \inf_w \left[\frac{\varepsilon}{2}\int_{\mathcal{O}} w^2 \, \mathrm{d}x + \phi(h_\varepsilon, s+\varepsilon)\right] + \text{`negligible terms'}. \tag{5.8}$$

Taking now the ε-expansion of the function $\phi(h_\varepsilon, s+\varepsilon)$ we obtain

$$\begin{aligned} \phi(h_\varepsilon, s+\varepsilon) &= \phi(h - \varepsilon A h + \varepsilon w \chi_{\mathcal{O}}, s+\varepsilon), \\ &= \phi(h,x) - \varepsilon \left(\frac{\partial \phi}{\partial h}(h,s), Ah\right) + \varepsilon \left(\frac{\partial \phi}{\partial h}(h,s), w\chi_{\mathcal{O}}\right) \\ &\quad + \varepsilon \frac{\partial \phi}{\partial s}(h,s) + \text{ higher-order terms,} \end{aligned} \tag{5.9}$$

where

$$\left(\frac{\partial \phi}{\partial h}(h,s), \hat{h}\right) = \lim_{\lambda \to 0} \frac{\mathrm{d}}{\mathrm{d}\lambda}\phi(h + \lambda \hat{h}, s),$$

with h and $\hat{h}$ in $L^2(\Omega)$ and, actually, smooth enough so that h and $\hat{h}$ belong to $H^2(\Omega) \cap H_0^1(\Omega)$.

Combining (5.8) to (5.9), dividing by ε, and letting $\varepsilon \to 0$, we obtain

$$\inf_w \left[\frac{1}{2}\int_{\mathcal{O}} w^2 \, \mathrm{d}x - \left(\frac{\partial \phi}{\partial h}(h,s), Ah\right) + \left(\frac{\partial \phi}{\partial h}(h,s), w\chi_{\mathcal{O}}\right) + \frac{\partial \phi}{\partial s}(h,s)\right] = 0; \tag{5.10}$$

hence it follows that

$$-\frac{\partial \phi}{\partial s}(h,s) + \left(\frac{\partial \phi}{\partial h}(h,s), Ah\right) + \frac{1}{2}\int_{\mathcal{O}} \left(\frac{\partial \phi}{\partial h}(h,s)\right)^2 \mathrm{d}x = 0. \tag{5.11}$$

The functional equation (5.11) is the *Hamilton–Jacobi–Bellmann* equation. It is a *partial differential equation in infinite dimensions* since $h \in L^2(\Omega)$ (in fact, $h \in H^2(\Omega) \cap H_0^1(\Omega)$), and where $s \in (0,T)$.

We have to add an *'initial' condition*, here for $t = T$, since we integrate (5.11) *backward in time*.

When $s \to T$, we have less and less time to 'correct' the trajectory. Therefore (this is again formal but it can be made precise without difficulty)

$$\phi(h,T) = \begin{cases} 0 \text{ if } h \in y_T + \beta B, \\ +\infty \text{ otherwise.} \end{cases} \tag{5.12}$$

5.3. Some remarks

Remark 5.1 The 'solution' of equations (5.11) and (5.12) should be defined in the framework of the *viscosity solutions* of Crandall and P.L. Lions (1985; 1986a,b; 1990; 1991), which was generalized by those authors to the infinite-dimensional case, which is the present situation.

Remark 5.2 Let h be given in $L^2(\Omega)$ and let y_h be the solution of

$$\frac{\partial y_h}{\partial t} + Ay_h = 0 \text{ in } \Omega \times (s,T), \ y_h(s) = h, \ y_h = 0 \text{ on } \Sigma_s \tag{5.13}$$

(i.e. we choose $v = 0$ in (5.1)–(5.3)). Let us denote by E_s the set of those functions h in (5.13) such that

$$y_h(T) \in y_T + \beta B. \tag{5.14}$$

We clearly have (from (5.11), (5.12))

$$\phi(h,s) = 0 \text{ if } h \in E_s. \tag{5.15}$$

We can – formally – draw the picture of Figure 43.

Remark 5.3 As usual in the *dynamic programming* approach, the *best decision* at time s corresponds to the element w in $L^2(\mathcal{O})$ which achieves the minimum in (5.10), namely

$$u(s) = -\frac{\partial \phi}{\partial h}(h,s)\chi_{\mathcal{O}}; \tag{5.16}$$

This is the 'real time' optimal policy – provided we know how to compute $(\partial\phi/\partial h)(h,s)$ – a formidable task indeed!

Remark 5.4 The Duality formulas of Section 1.4 can of course be applied. We obtain

$$\phi(h,s) = -\inf_{\hat{f}\in L^2(\Omega)} \left[\frac{1}{2}\iint_{\mathcal{O}\times(s,T)} \hat{\psi}^2 \,dx\,dt - \int_\Omega \hat{f}(y_T - y_h(T))\,dx + \beta\|\hat{f}\|_{L^2(\Omega)}\right], \tag{5.17}$$

where y_h is defined by (5.13) and where $\hat{\psi}$ is defined by

$$-\frac{\partial\hat{\psi}}{\partial t} + A^*\hat{\psi} = 0 \text{ in } \Omega \times (s,T), \ \hat{\psi}(T) = \hat{f}, \ \hat{\psi} = 0 \text{ on } \Sigma_s. \tag{5.18}$$

Remark 5.5 *Dynamic programming* has been applied to the *closed loop* control of the *Navier–Stokes equations for incompressible viscous flow* in Sritharan (1991a,b).

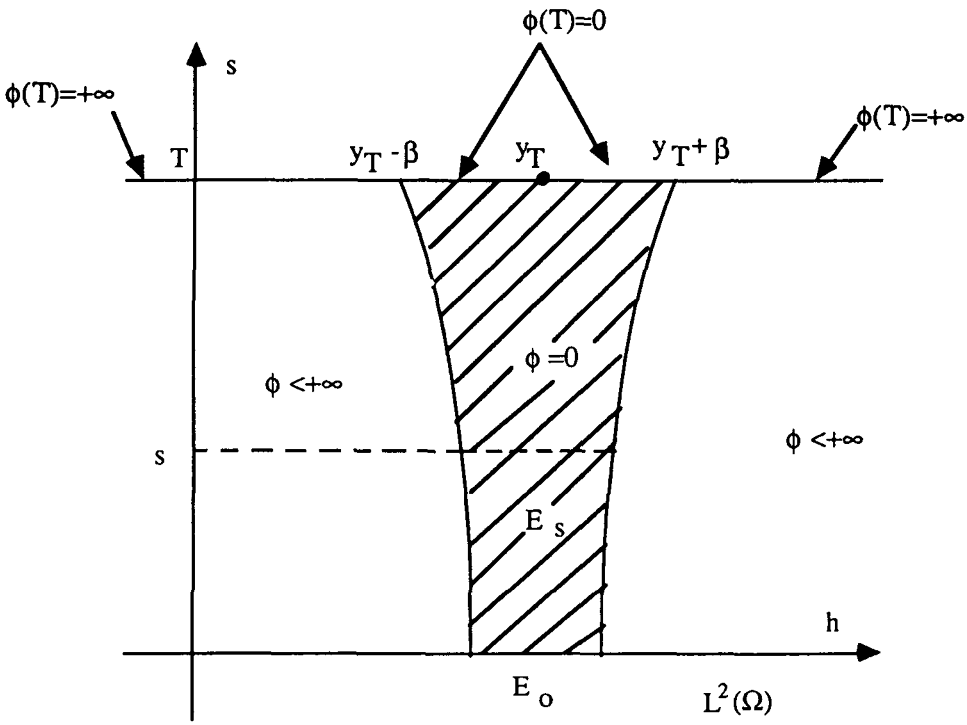


Fig. 43. Distribution of ϕ in the set $L^2(\Omega) \times (0,T)$.

6. WAVE EQUATIONS

6.1. Wave equations: Dirichlet boundary control

Let Ω be a *bounded* open set in $\mathbb{R}^d$, with a smooth boundary Γ. In $Q = \Omega \times (0,T)$, we consider the *wave equation*

$$\frac{\partial^2 y}{\partial t^2} + Ay = 0, \tag{6.1}$$

where A is a *second-order elliptic operator*, with *smooth coefficients*, and such that

$$A = A^*. \tag{6.2}$$

A classical case is

$$A = -\Delta\Big(= -\nabla^2 = -\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}\Big). \tag{6.3}$$

We assume that

$$y(0) = 0, \quad \frac{\partial y}{\partial t}(0) = 0, \tag{6.4}$$

and we suppose that the *control is applied on a part of the boundary.* More precisely, let Γ_0 be a 'smooth' subset of Γ. Then

$$y = \begin{cases} v \text{ on } \Sigma_0 = \Gamma_0 \times (0,T), \\ 0 \text{ on } \Sigma \backslash \Sigma_0, \Sigma = \Gamma \times (0,T). \end{cases} \tag{6.5}$$

We denote by $y(v) : t \to y(t;v)$ the solution of the wave problem (6.1), (6.4), (6.5), assuming that the control v satisfies 'some' further properties. Indeed, we shall assume that

$$v \in L^2(\Sigma_0), \tag{6.6}$$

since this is – as far as the control itself is concerned! – certainly the simplest possible choice. However, a few preliminary remarks are necessary here.

Remark 6.1. Even assuming that Γ, Γ_0 and the coefficients of operator A are very smooth, once the choice (6.6) has been made, one *has* to deal with *weak solutions* of (6.1), (6.4), (6.5). In fact (cf. Lions (1988a) and (1988b, Vol. 1)) the (unique) solution $y(v)$ of (6.1), (6.4), (6.5) satisfies the following properties

$$y(v) \text{ is continuous from } [0,T] \text{ to } L^2(\Omega), \tag{6.7}$$

$$y_t(v) \text{ is continuous from } [0,T] \text{ to } H^{-1}(\Omega), \tag{6.8}$$

where, in (6.8) and in the following, we have set

$$\varphi_t = \frac{\partial \varphi}{\partial t}, \quad \varphi_{tt} = \frac{\partial^2 \varphi}{\partial t^2}.$$

The solution $y = y(v)$ is defined by *transposition* as in Lions and Magenes (1968). If we consider the adjoint equation

$$\begin{cases} \varphi_{tt} + A\varphi = f \text{ in } Q, \\ \varphi(T) = \varphi_t(T) = 0, \quad \varphi = 0 \text{ on } \Sigma, \end{cases} \tag{6.9}$$

where $f \in L^1(0,T;L^2(\Omega))$, then y is defined by

$$\int_Q yf \,\mathrm{d}x\,\mathrm{d}t = -\int_{\Sigma_0} \frac{\partial \varphi}{\partial n_A} v \,\mathrm{d}\Sigma, \tag{6.10}$$

where $\partial/\partial n_A$ denotes the *normal derivative* associated with A (it is the usual normal derivative if $A = -\Delta$). The linear form

$$f \to -\int_{\Sigma_0} \frac{\partial \varphi}{\partial n_A} v \,\mathrm{d}\Sigma$$

is *continuous* over $L^1(0,T;L^2(\Omega))$; this is the *key point* since we can show that

$$\left\| \frac{\partial \varphi}{\partial n_A} \right\|_{L^2(\Sigma)} \leq C\|f\|_{L^1(0,T;L^2(\Omega))}. \tag{6.11}$$

One uses then the restriction of $\partial\varphi/\partial n_A$ to Σ_0 and therefore

$$y \in L^\infty(0, T; L^2(\Omega)).$$

One proceeds then to obtain (6.7), (6.8).

Remark 6.2. The original proof (Lions (1983)) assumes that Γ is *smooth.* Strangely enough, it took ten years – and a nontrivial technical proof – to generalize (6.11) to *Lipschitz boundaries* (in the sense of Nečas (1967)); this was done by Chiara (1993).

We now want to study the *controllability* for systems modelled by (6.1), (6.4), (6.5), i.e., *given*

$$T(0 < T < +\infty), \quad \textit{given } \{z^0, z^1\} \in L^2(\Omega) \times H^{-1}(\Omega),$$

can we find v *such that*

$$\begin{cases} y(T; v) = z^0 \text{ or } y(T; v) \text{ `very close' to } z^0, \\ y_t(T; v) = z^1 \text{ or } y_t(T; v)\text{`very close' to } z^1. \end{cases} \tag{6.12}$$

There is a *fundamental difference* between the present situation and those discussed in Sections 1 and 2 for *diffusion* equations, due to the *finite propagation velocity* of the waves (or singularities) the solution is made of, whereas this velocity is *infinite* for *diffusion* equations (and for *Petrowsky's* type equations as well). It follows from this property that

$$\text{Conditions (6.12) may be possible only if } T \text{ is sufficiently large.} \tag{6.13}$$

This will be made precise in the following sections.

6.2. Approximate controllability

For technical reasons, we shall always consider the mapping

$$L : v \to \{-y_t(T; v), y(T; v)\} \tag{6.14}$$

(which is a *continuous linear* mapping from $L^2(\Sigma_0)$ into $H^{-1}(\Omega) \times L^2(\Omega)$), instead of the mapping

$$v \to \{y(T; v), y_t(T; v)\}$$

(but this does not go beyond simplifying – we hope – some formulae).

Let us first discuss the range $R(L)$ of operator L; we consider thus $\mathbf{f} = \{f^0, f^1\}$ such that

$$\mathbf{f} \in H_0^1(\Omega) \times L^2(\Omega) \tag{6.15}$$

and

$$\langle Lv, \mathbf{f}\rangle = 0, \quad \forall v \in L^2(\Sigma_0), \tag{6.16}$$

i.e.

$$-\langle y_t(T;v), f^0\rangle + \int_\Omega y(T;v) f^1 \, \mathrm{d}x = 0, \ \forall v \in L^2(\Sigma_0); \tag{6.16$'$}$$

in (6.16) (respectively (6.16)$'$), $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega) \times L^2(\Omega)$ and $H_0^1(\Omega) \times L^2(\Omega)$ (respectively $H^{-1}(\Omega)$ and $H_0^1(\Omega)$).

We introduce ψ solution of

$$\psi_{tt} + A\psi = 0 \text{ in } Q = \Omega \times (0,T), \quad \psi(T) = f^0, \ \psi_t(T) = f^1, \ \psi = 0 \text{ on } \Sigma. \tag{6.17}$$

It is a smooth solution, which satisfies in particular

$$\frac{\partial \psi}{\partial n_A} \in L^2(\Sigma), \left\| \frac{\partial \psi}{\partial n_A} \right\|_{L^2(\Sigma)} \leq C(\|f^0\|_{H_0^1(\Omega)} + \|f^1\|_{L^2(\Omega)}). \tag{6.18}$$

Multiplying the first equation in (6.17) by y and integrating by parts, we obtain

$$\langle Lv, \mathbf{f} \rangle = \int_{\Sigma_0} \frac{\partial \psi}{\partial n_A} v \, \mathrm{d}\Sigma. \tag{6.19}$$

Thus (6.16) is equivalent to

$$\frac{\partial \psi}{\partial n_A} = 0 \text{ on } \Sigma_0. \tag{6.20}$$

Therefore the *Cauchy data* are zero for ψ on Σ_0. According to the *Holmgren's Uniqueness Theorem* (cf. Hörmander (1976)) it follows that

$$\begin{gathered}\text{If } T > 2(\text{diameter of } \Omega), \text{ then } \{y(T;v), y_t(T;v)\} \\ \text{describes a dense subspace of } L^2(\Omega) \times H^{-1}(\Omega)\end{gathered} \tag{6.21}$$

(in (6.21), the 'diameter' of Ω is related to the geodetic distance associated with A. It is the usual geodetic distance if $A = -\Delta$).

Indeed, according to *Holmgren's Theorem*, we have $\psi \equiv 0$ so that $f = 0$ (see, for example, Lions (1988b, Vol. 1)).

Remark 6.3. Holmgren's theorem applies with the conditions

$$\psi = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial n_A} = 0 \text{ on } \Sigma_0,$$

without having necessarily $\psi = 0$ on $\Sigma \backslash \Sigma_0$. The fact that in the present situation we have $\psi = 0$ on the whole Σ provides some more flexibility to obtain uniqueness results. We shall return to this later on.

6.3. Formulation of the approximate controllability problem

We shall make the following hypothesis:

$$\Sigma_0 \text{ allows the application of the Holmgren's Uniqueness Theorem.} \tag{6.22}$$

Then, $\{z^0, z^1\}$ being given in $L^2(\Omega) \times H^{-1}(\Omega)$, there always exist controls v (actually an infinite number of them) such that

$$y(T; v) \in z^0 + \beta_0 B, \quad y_t(T; v) \in z^1 + \beta_1 B_{-1}, \tag{6.23}$$

where B (respectively B_{-1}) denotes the unit ball of $L^2(\Omega)$ (respectively $H^{-1}(\Omega)$), and where β_0, β_1 are given positive members, arbitrarily small.

The *optimal control* problem that we consider is

$$\inf_v \frac{1}{2} \int_{\Sigma_0} v^2 \,\mathrm{d}\Sigma, \quad v \text{ satisfying (6.23)}. \tag{6.24}$$

Remark 6.4. *Exact controllability* corresponds to $\beta_0 = \beta_1 = 0$.

6.4. Dual problems

We proceed essentially as in Section 1.4. We introduce therefore

$$F_1(v) = \frac{1}{2} \int_{\Sigma_0} v^2 \,\mathrm{d}\Sigma, \ \forall v \in L^2(\Sigma_0), \tag{6.25}$$

and then $F_2 : H^{-1}(\Omega) \times L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$F_2(\hat{\mathbf{f}}) = F_2(\hat{f}^0, \hat{f}^1) = \begin{cases} 0 \text{ if } \hat{f}^0 \in -z^1 + \beta_1 B_{-1} \text{ and } \hat{f}^1 \in z^0 + \beta_0 B, \\ +\infty \text{ otherwise.} \end{cases} \tag{6.26}$$

With this notation, the control problem (6.20) can be formulated as

$$\inf_{v \in L^2(\Sigma_0)} [F_1(v) + F_2(Lv)]. \tag{6.27}$$

Using, as in Section 1.4, *Duality Theory* we obtain

$$\int_{v \in L^2(\Sigma_0)} [F_1(v) + F_2(Lv)] + \inf_{\hat{\mathbf{f}} \in H_0^1(\Omega) \times L^2(\Omega)} [F_1^*(L^*\hat{\mathbf{f}}) + F_2^*(-\hat{\mathbf{f}})] = 0, \tag{6.28}$$

where

$$F_1^*(v) = \frac{1}{2} \int_{\Sigma_0} v^2 \,\mathrm{d}\Sigma, \ \forall v \in L^2(\Sigma_0), \tag{6.29$_1$}$$

$$\begin{cases} F_2^*(\hat{\mathbf{f}}) = -\langle z^1, \hat{f}^0 \rangle + \int_\Omega z^0 \hat{f}^1 \,\mathrm{d}x + \beta_1 \|\hat{f}^0\|_{H_0^1(\Omega)} + \beta_0 \|\hat{f}^1\|_{L^2(\Omega)}, \\ \forall \hat{\mathbf{f}} = \{\hat{f}^0, \hat{f}^1\} \in H_0^1(\Omega) \times L^2(\Omega). \end{cases} \tag{6.29$_2$}$$

Using (6.19) we have

$$L^*\hat{\mathbf{f}} = \frac{\partial \hat{\psi}}{\partial n_A} \text{ on } \Sigma_0, \tag{6.30}$$

where $\hat{\psi}$ is given by (6.17) (with $\mathbf{f} = \hat{\mathbf{f}}$). We have therefore the following

Theorem 6.1 *We suppose that* (6.22) *holds true. For* β_0 *and* β_1 *given arbitrarily small, problem* (6.24) *has a unique solution such that*

$$\inf_v \frac{1}{2}\int_{\Sigma_0} v^2 \,\mathrm{d}\Sigma = -\inf_{\hat{\mathbf{f}}} \left[\frac{1}{2}\int_{\Sigma_0}\left(\frac{\partial\hat{\psi}}{\partial n_A}\right)^2 \mathrm{d}\Sigma + \langle z^1, \hat{f}^0\rangle \right.$$
$$\left. - \int_\Omega z^0 \hat{f}^1 \,\mathrm{d}x + \beta_1 \|\hat{f}^0\|_{H_0^1(\Omega)} + \beta_0 \|\hat{f}^1\|_{L^2(\Omega)}\right], \tag{6.31}$$

where, in (6.31), $v \in L^2(\Sigma_0)$ *and verifies* (6.23), $\hat{\mathbf{f}} \in H_0^1(\Omega) \times L^2(\Omega)$, *and where* $\hat{\psi}$ *is given by* (6.17), *with* $\mathbf{f} = \hat{\mathbf{f}}$.

The *dual problem* is the minimization problem in the right-hand side of (6.31). If $\mathbf{f}$ is the solution of the dual problem and if ψ is the corresponding solution of (6.17) then the optimal control, i.e. the solution u of problem (6.24) is given by

$$u = \frac{\partial\psi}{\partial n_A} \text{ on } \Sigma_0. \tag{6.32}$$

6.5. Direct solution of the dual problem

One can formulate the dual problem in an equivalent fashion which will be useful when β_0 and β_1 converge to zero, and also for numerical calculations.

To this effect, we introduce the following operator Λ:

Given $\hat{\mathbf{f}} = \{\hat{f}^0, \hat{f}^1\} \in H_0^1(\Omega) \times L^2(\Omega)$, we define $\hat{\psi}$ and $\hat{y}$ by

$$\hat{\psi}_{tt} + A\hat{\psi} = 0 \text{ in } Q, \quad \hat{\psi}(T) = \hat{f}^0,\ \hat{\psi}_t(T) = \hat{f}^1,\ \hat{\psi} = 0 \text{ on } \Sigma, \tag*{$(6.33)_1$}$$

$$\hat{y}_{tt} + A\hat{y} = 0 \text{ in } Q, \quad \hat{y}(0) = \hat{y}_t(0) = 0,\ \hat{y} = \frac{\partial\hat{\psi}}{\partial n_A} \text{ on } \Sigma_0,\ \hat{y} = 0 \text{ on } \Sigma\backslash\Sigma_0, \tag*{$(6.33)_2$}$$

and we set

$$\Lambda\hat{\mathbf{f}} = \{-\hat{y}_t(T), \hat{y}(T)\}. \tag{6.34}$$

We define in this way an operator Λ such that

$$\Lambda \in \mathcal{L}(H_0^1(\Omega) \times L^2(\Omega); H^{-1}(\Omega) \times L^2(\Omega)). \tag{6.35}$$

If we multiply both sides of the first equation in $(6.33)_2$ by $\hat{\psi}'$ (which corresponds to $\hat{\mathbf{f}}' \in H_0^1(\Omega) \times L^2(\Omega)$) and if we integrate by parts we obtain (with obvious notation):

$$\langle \Lambda\hat{\mathbf{f}}, \hat{\mathbf{f}}'\rangle = \int_{\Sigma_0} \frac{\partial\hat{\psi}}{\partial n_A}\frac{\partial\hat{\psi}'}{\partial n_A}\,\mathrm{d}\Sigma. \tag{6.36}$$

It follows from (6.36) that the operator Λ is *self-adjoint* and *positive semi-definite.*

The *dual problem* is then *equivalent* to

$$\inf_{\hat{\mathbf{f}}} \left[\frac{1}{2}\langle \Lambda \hat{\mathbf{f}}, \hat{\mathbf{f}} \rangle + \langle z^1, \hat{f}^0 \rangle - \int_\Omega z^0 \hat{f}^1 \, \mathrm{d}x + \beta_1 \|\hat{f}^0\|_{H_0^1(\Omega)} + \beta_0 \|\hat{f}^1\|_{L^2(\Omega)} \right], \quad (6.37)$$

where, in (6.37), $\hat{\mathbf{f}} \in H_0^1(\Omega) \times L^2(\Omega)$.

Assuming that the condition (6.22) holds true, problem (6.37) has a *unique* solution for β_0 and $\beta_1 > 0$, arbitrarily small. If we denote by $\mathbf{f}$ the solution of problem (6.37) it is also the solution of the following *variational inequality*

$$\begin{aligned} &\mathbf{f} \in H_0^1(\Omega) \times L^2(\Omega); \ \forall \hat{\mathbf{f}} \in H_0^1(\Omega) \times L^2(\Omega) \text{ we have} \\ &\langle \Lambda \mathbf{f}, \hat{\mathbf{f}} - \mathbf{f} \rangle + \langle z^1, \hat{f}^0 - f^0 \rangle - \int_\Omega z^0 (\hat{f}^1 - f^1) \, \mathrm{d}x \\ &+\beta_1 (\|\hat{f}^0\|_{H_0^1(\Omega)} - \|f^0\|_{H_0^1(\Omega)}) \\ &+\beta_0 (\|\hat{f}^1\|_{L^2(\Omega)} - \|f^1\|_{L^2(\Omega)}) \geq 0. \end{aligned} \quad (6.38)$$

Remark 6.5. Problems (6.37), (6.38) are equivalent to the minimization problem in the right-hand side of (6.31), but they are better suited for the solution of the dual problem.

Remark 6.6. The operator Λ is the same as the one introduced in the *Hilbert Uniqueness Method* (HUM). This is made more precise in the following section (see also Lions (1986; 1988a,b)).

Remark 6.7. Relation (6.36) makes sense, because there exists a constant C such that

$$\int_{\Sigma_0} \left| \frac{\partial \hat{\psi}}{\partial n_A} \right|^2 \mathrm{d}\Sigma \leq C(\|\hat{f}^0\|^2_{H_0^1(\Omega)} + \|\hat{f}^1\|_{L^2(\Omega)}), \quad (6.39)$$

where, in (6.39), $\hat{\psi}$ and $\hat{\mathbf{f}} = \{\hat{f}^0, \hat{f}^1\}$ are related by (6.33).

6.6. Exact controllability and new functional spaces

Let us now consider problem (6.37), (6.38) with the idea of letting β_0 and β_1 converge to zero. We introduce on $H_0^1(\Omega) \times L^2(\Omega)$ the following new functional

$$[\hat{f}] = (\langle \Lambda \hat{f}, \hat{f} \rangle)^{1/2}. \quad (6.40)$$

Since we assume that the condition (6.22) holds true, the functional $[\cdot]$ is in fact a *norm*, of a *pre-Hilbertian* nature. We introduce then

$$E = \text{ Completion of } H_0^1(\Omega) \times L^2(\Omega) \text{ for the norm } [\cdot]; \quad (6.41)$$

with this notation we can state that

$$\Lambda \text{ is an isomorphism from } E \text{ onto } E'. \quad (6.42)$$

If $\beta_0 = \beta_1 = 0$, problem (6.37), (6.38) is *equivalent* to

$$\inf_{\hat{\mathbf{f}}} \left[\frac{1}{2}[\hat{f}]^2 + \langle z^1, \hat{f}^0 \rangle - \int_\Omega z^0 \hat{f}^1 \, \mathrm{d}x \right], \quad \hat{\mathbf{f}} \in H_0^1(\Omega) \times L^2(\Omega). \tag{6.43}$$

Problem (6.44) *has a unique solution if and only if*

$$\{-z^1, z^0\} \in E'. \tag{6.44}$$

If we denote by $\mathbf{f}_\beta = \{f_\beta^0, f_\beta^1\}$ the solution of problem (6.37), (6.38), where $\beta = \{\beta_0, \beta_1\}$, then

$$\lim_{\beta \to 0} \mathbf{f}_\beta = \mathbf{f}_0 = \text{ the solution of (6.43)} \tag{6.45}$$

if and only if condition (6.44) holds true.

Remark 6.8. The method of solution that we have just presented is what is called HUM (*Hilbert Uniqueness Method*) (cf. Lions (1986; 1988a,b)) since the key element is the introduction of the new Hilbert space E based on a *uniqueness* property.

Remark 6.9. Problems (6.38) or (6.43) give a *constructive* approach to approximate or exact controllability; we shall make this more precise in the next sections.

Remark 6.10. Condition (6.44) means that *exact controllability is possible if and only if z^0 and z^1 are taken in a convenient Hilbert space.*

Remark 6.11. The approach taken in the present section is closely related to the one followed in Section 1.5. With the notation of Section 1.5, Remark 1.14, we would have

$$E = \widehat{H_0^1(\Omega) \times L^2(\Omega)}.$$

There is, however, a very important technical difference between the two situations, since for the *diffusion* problems discussed in Section 1 the space $\widehat{L^2(\Omega)}$ is *never* a 'simple' *distribution space* (except for the case $\mathcal{O} = \Omega$, i.e. the control is distributed over the whole domain Ω). For the *wave equation* the situation is quite different, as we shall see in the next section.

6.7. On the structure of space E

We follow here Bardos, Lebeau and Rauch (1988).

We shall say that Σ_0 enjoys the *geometrical control condition* if any ray, starting from any point of Ω at $t = 0$, reaches eventually (after geometrical reflexions on Γ) the set Γ_0 before time $t = T$.

The main result is then

$$\text{If } \Sigma_0 \text{ satisfies the geometrical control condition, then } E = H_0^1(\Omega) \times L^2(\Omega). \tag{6.46}$$

Actually the geometrical control condition is also *necessary* in order (6.46) to be true. The inequality corresponding to (6.46) is the *reverse* of inequality (6.39): there exists a constant $C_1 > 0$ such that

$$\int_{\Sigma_0} \left| \frac{\partial \hat{\psi}}{\partial n_A} \right|^2 \mathrm{d}\Sigma \geq C_1(\|\hat{f}^0\|^2_{H^1_0(\Omega)} + \|\hat{f}^1\|^2_{L^2(\Omega)}) \tag{6.47}$$

if and only if Σ_0 satisfies the geometrical control condition.

Remark 6.12. We refer to Lions (1988b) for the various contributions, by many authors, which led to the fundamental inequality (6.47).

Remark 6.13. If Σ_0 *does not* satisfy the geometrical control condition, but *does* satisfy the conditions for the Holmgren's Uniqueness Theorem, then

$$[\hat{f}] = \left(\int_{\Sigma_0} \left| \frac{\partial \hat{\psi}}{\partial n_A} \right|^2 \mathrm{d}\Sigma \right)^{1/2}$$

is a norm, *strictly weaker* than the $H^1_0(\Omega) \times L^2(\Omega)$ norm. In that case E is a *new* Hilbert space, such that

$$H^1_0(\Omega) \times L^2(\Omega) \subset E, \quad \text{strictly}, \tag{6.48}$$

and the exact structure of E is far from being simple, since the space E *can contain elements which are not distributions on* Ω.

6.8. Numerical methods for the Dirichlet boundary controllability of the wave equation

6.8.1. Generalities. Synopsis

In this section which is largely inspired by Dean, Glowinski and Li (1989), Glowinski *et al.* (1990), Glowinski (1992a) we shall discuss the *numerical solution* of the *exact* and *approximate Dirichlet boundary controllability problems* considered in the preceding sections.

To make it simpler we shall assume that Σ_0 satisfies the *geometrical control condition* (see the above section), so that

$$E = H^1_0(\Omega) \times L^2(\Omega), \tag{6.49}$$

and the operator Λ defined in Section 6.5 is an *isomorphism* from E onto $E'(= H^{-1}(\Omega) \times L^2(\Omega))$. The properties of Λ (*symmetry* and *strong ellipticity*) will make the solution of the exact controllability problem possible by a *conjugate gradient algorithm* operating in the space E. We shall describe next the *time* and *space discretizations* of the exact controllability problem by a combination of *finite difference* (FD) and *finite element* (FE) methods and then discuss the iterative solution of the corresponding approximate problem. Finally we shall describe solution methods for the *approximate* boundary controllability problem (6.24).

Both exact and approximate controllability problems will be solved using their *dual formulation* since the corresponding control problems are *easier* to solve than their primal counterparts.

The results of numerical experiments obtained by the methods described in the present section will be reported in Section 6.9, hereafter.

Remark 6.14. A *spectral* method – still based on HUM – for solving *directly* (i.e. non*iteratively*) the *exact* Dirichlet boundary controllability problem is discussed in Bourquin (1993), where numerical results are also presented.

6.8.2. Dual formulation of the exact controllability problem. Further properties of Λ

To obtain the *dual problem* corresponding to *exact controllability* it suffices to take $\beta_0 = \beta_1 = 0$ in formulations (6.37), (6.38); we obtain then

$$\Lambda \mathbf{f} = \{-z^1, z^0\}. \tag{6.50}$$

Since we supposed (see Section 6.8.1) that the *geometrical control condition holds*, we know (from Section 6.6) that

$$\Lambda \text{ is an isomorphism from } E \text{ onto } E', \tag{6.51}$$

with $E = H_0^1(\Omega) \times L^2(\Omega), E' = H^{-1}(\Omega) \times L^2(\Omega)$. Problem (6.50) has therefore a *unique* solution, $\forall \{z^0, z^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$. The solution $\mathbf{f}$ of (6.50) is also *the* solution of the following *linear variational* problem

$$\begin{cases} \mathbf{f} \in E;\ \forall \hat{\mathbf{f}} = \{\hat{f}^0, \hat{f}^1\} \in E \quad \text{we have} \\ \langle \Lambda \mathbf{f}, \hat{\mathbf{f}} \rangle = -\langle z^1, \hat{f}^0 \rangle + \int_\Omega z^0 \hat{f}^1 \, dx. \end{cases} \tag{6.52}$$

Since (from Sections 6.5 to 6.7) Λ is *continuous, self-adjoint* and *strongly elliptic* (in the sense that there exists $C > 0$ such that

$$\langle \Lambda \hat{\mathbf{f}}, \hat{\mathbf{f}} \rangle \geq C \|\hat{\mathbf{f}}\|_E^2, \quad \forall \hat{\mathbf{f}} \in E)$$

the *bilinear functional*

$$\{\hat{\mathbf{f}}, \hat{\mathbf{f}}'\} \to \langle \Lambda \hat{\mathbf{f}}, \hat{\mathbf{f}}' \rangle : E \times E \to \mathbb{R}$$

is *continuous, symmetric* and *E-elliptic* over $E \times E$. On the other hand, the *linear* functional in the right-hand side of (6.52) is clearly *continuous* over E, implying (cf. Section 1.8.2) that problem (6.50), (6.52) can be solved by a *conjugate gradient algorithm* operating in the space E. such an algorithm will be described in the following section.

Remark 6.15. We suppose here that $\Gamma_0 = \Gamma$ and that $A = -\Delta$; we suppose also that there exists $x_0 \in \Omega$ and $C > 0$ such that

$$\overrightarrow{x_0 M} \cdot \mathbf{n} = C, \quad \forall M \in \Gamma, \tag{6.53}$$

with $\mathbf{n}$ the unit vector of the outward normal at Γ, at M. Domains satisfying

(6.53) are easy to characterize geometrically, simple cases being disks and squares. Now let us denote by Λ_T the operator Λ associated with T. It has been shown by J.L. Lions (unpublished result) and Bensoussan (1990) that

$$\lim_{T\to+\infty} \frac{\Lambda_T}{T} = \frac{1}{C}\begin{bmatrix} -\Delta & 0 \\ 0 & I \end{bmatrix}. \tag{6.54}$$

Result (6.54) is quite important for the validation of the numerical methods described hereafter, since it easily provides

$$\lim_{T\to+\infty} T\mathbf{f}_T = \{\chi^0, \chi^1\}, \tag{6.55}$$

where, from (6.54),

$$\Delta\chi^0 = Cz^1 \text{ in } \Omega, \quad \chi^0 = 0 \text{ on } \Gamma, \tag{6.56}$$

$$\chi^1 = Cz^0. \tag{6.57}$$

6.8.3. Conjugate gradient solution of problem (6.50), (6.52).

Assuming that the *geometrical control condition* holds, it follows from Section 6.8.2. that we can apply the general *conjugate gradient* algorithm (1.122)–(1.129) to the solution of problem (6.50), (6.52); indeed, it suffices to take

$$V = E, a(\cdot,\cdot) = \langle\Lambda\cdot,\cdot\rangle,\ L : \hat{\mathbf{f}} \to -\langle z^1, \hat{f}^0\rangle + \int_\Omega z^0 \hat{f}^1 \, dx.$$

On E, we shall use as scalar product

$$\{\mathbf{v},\mathbf{w}\} \to \int_\Omega (\nabla v^0 \cdot \nabla w^0 + v^1 w^1)\, dx, \quad \forall \mathbf{v},\mathbf{w} \in E.$$

We obtain then the following algorithm

Algorithm. Step 0: Initialization

$$f_0^0 \in H_0^1(\Omega) \quad \text{and} \quad f_0^1 \in L^2(\Omega) \text{ are given;} \tag{6.58}$$

solve then

$$\begin{cases} \dfrac{\partial^2\psi_0}{\partial t^2} + A\psi_0 = 0 \text{ in } Q, & \psi_0 = 0 \text{ on } \Sigma, \\ \psi_0(T) = f_0^0, & \dfrac{\partial\psi_0}{\partial t}(T) = f_0^1, \end{cases} \tag{6.59}$$

and

$$\begin{cases} \dfrac{\partial^2\varphi_0}{\partial t^2} + A\varphi_0 = \text{ in } Q, \quad \varphi_0 = \dfrac{\partial\psi_0}{\partial n_A} \text{ on } \Sigma_0, \quad \varphi_0 = 0 \text{ on } \Sigma\backslash\Sigma_0, \\ \varphi_0(0) = 0, \quad \dfrac{\partial\varphi_0}{\partial t}(0) = 0. \end{cases} \tag{6.60}$$

Compute $\mathbf{g}_0 = \{g_0^0, g_0^1\} \in E$ *by*

$$-\Delta g_0^0 = z^1 - \frac{\partial\varphi_0}{\partial t}(T) \text{ in } \Omega, g_0^0 = 0 \text{ on } \Gamma, \tag{6.61}$$

$$g_0^1 = \varphi_0(T) - z^0, \tag{6.62}$$

respectively. Set then

$$\mathbf{w}_0 = \mathbf{g}_0. \tag{6.63}$$

Now, for $n \geq 0$, assuming that $\mathbf{f}_n, \mathbf{g}_n, \mathbf{w}_n$ are known, compute $\mathbf{f}_{n+1}$, $\mathbf{g}_{n+1}$, $\mathbf{w}_{n+1}$ as follows.

Step 1: Descent

Solve

$$\begin{cases} \dfrac{\partial^2 \bar{\psi}_n}{\partial t^2} + A\bar{\psi}_n = 0 \text{ in } Q, \quad \bar{\psi}_n = 0 \text{ on } \Sigma, \\ \bar{\psi}_n(T) = w_n^0, \quad \dfrac{\partial \bar{\psi}_n}{\partial t}(T) = w_n^1, \end{cases} \tag{6.64}$$

$$\begin{cases} \dfrac{\partial^2 \bar{\varphi}_n}{\partial t^2} + A\bar{\varphi}_n = 0 \text{ in } Q, \quad \bar{\varphi}_n = \dfrac{\partial \bar{\psi}^n}{\partial n_A} \text{ on } \Sigma_0, \quad \bar{\varphi}_n = 0 \text{ on } \Sigma\backslash\Sigma_0, \\ \bar{\varphi}_n(0) = 0, \quad \dfrac{\partial \bar{\varphi}_n}{\partial t}(0) = 0, \end{cases} \tag{6.65}$$

$$\Delta \bar{g}_n^0 = \frac{\partial \bar{\varphi}_n}{\partial t}(T) \text{ in } \Omega, \bar{g}_n^0 = 0 \text{ on } \Gamma, \tag{6.66}$$

and set

$$\bar{g}_n^1 = \bar{\varphi}_n(T). \tag{6.67}$$

Compute now

$$\rho_n = \int_\Omega (|\nabla g_n^0|^2 + |g_n^1|^2)\, \mathrm{d}x \Big/ \int_\Omega (\nabla \bar{g}_n^0 \cdot \nabla w_n^0 + \bar{g}_n^1 w_n^1)\, \mathrm{d}x, \tag{6.68}$$

$$\mathbf{f}_{n+1} = \mathbf{f}_n - \rho_n \mathbf{w}_n, \tag{6.69}$$

$$\mathbf{g}_{n+1} = \mathbf{g}_n - \rho_n \bar{\mathbf{g}}_n. \tag{6.70}$$

Step 2: Test of the convergence and construction of the new descent direction. If $\mathbf{g}_{n+1} = \mathbf{0}$, *or is sufficiently small (i.e.*

$$\int_\Omega (|\nabla g_{n+1}^0|^2 + |g_{n+1}^1|^2)\, \mathrm{d}x \Big/ \int_\Omega (|\nabla g_0^0|^2 + |g_0^1|^2)\, \mathrm{d}x \leq \epsilon^2) \tag{6.71}$$

take $\mathbf{f} = \mathbf{f}_{n+1}$; if not, compute

$$\gamma_n = \int_\Omega (|\nabla g_{n+1}^0|^2 + |g_{n+1}^1|^2)\, \mathrm{d}x \Big/ \int_\Omega (|\nabla g_n^0|^2 + |g_n^1|^2)\, \mathrm{d}x, \tag{6.72}$$

and set

$$\mathbf{w}_{n+1} = \mathbf{g}_{n+1} + \gamma_n \mathbf{w}_n. \tag{6.73}$$

Do $n = n + 1$ and go to (6.64).

Remark 6.16. It appears at first glance that algorithm (6.58)–(6.73) is quite memory demanding since it seems to require the storage of $\partial\bar{\psi}_n/\partial n_A|_{\Sigma_0}$ (in practice the storage of $\partial\bar{\psi}_n/\partial n_A$ over a discrete – but still large – subset of Σ_0). In fact, we can avoid this storage problem by observing that since the wave equation in (6.64) is *reversible* we can integrate *simultaneously, from* 0 to T, the wave equations (6.65) and

$$\begin{cases} \dfrac{\partial^2\bar{\psi}_n}{\partial t^2} + A\bar{\psi}_n = 0 \text{ in } Q, \quad \bar{\psi}_n = 0 \text{ on } \Sigma, \\ \bar{\psi}_n(0) \text{ and } \dfrac{\partial\bar{\psi}_n}{\partial t}(0) \text{ known from the integration of (6.64) from } T \text{ to } 0. \end{cases} \tag{6.74}$$

In the particular case where an *explicit scheme* is used for solving the wave equations (6.64), (6.65) and (6.74), the extra cost associated with the solution of (6.74) is *negligible* compared with the saving due to not storing $\partial\bar{\psi}_n/\partial n_A$ on Σ_0.

Remark 6.17. Once the solution $\mathbf{f}$ of the dual problem (6.50), (6.52) is known it suffices to integrate the wave equation $(6.33)_1$ with $\hat{\mathbf{f}} = \mathbf{f}$ to obtain ψ. The optimal control u, solution of the exact controllability problem is given then by

$$u = \left.\frac{\partial\psi}{\partial n_A}\right|_{\Sigma_0}. \tag{6.75}$$

6.8.4. Finite difference approximation of the dual problem (6.50), (6.52)

6.8.4.1. Generalities. An FE/FD approximation of problem (6.50), (6.52) will be discussed in Section 6.8.7 (see also Glowinski *et al.* (1990), Glowinski (1992a), and the references therein). At the present moment, we shall concentrate on the case where

$$\Omega = (0,1)^2, \quad A = -\Delta, \quad \Gamma_0 = \Gamma,$$

and where FD methods are used both for the space and time discretizations. Indeed, these approximations can also be obtained via space discretizations associated with FE grids like the one shown on Figure 1 of Section 2.6 (we should use, as shown in Glowinski *et al.* (1990), piecewise linear approximations and numerical integration by the trapezoidal rule).

Let I and N be positive integers; we define h (*space discretization step*) and Δt (*time discretization step*) by

$$h = \frac{1}{(I+1)}, \quad \Delta t = \frac{T}{N}, \tag{6.76}$$

respectively, and then denote by M_{ij} the point $\{ih, jh\}$.

6.8.4.2. Approximation of the wave equation $(6.33)_1$. Let us first discuss the discretization of the following wave problem

$$\begin{cases} \psi_{tt} - \Delta\psi = 0 \text{ in } Q, \quad \psi = 0 \text{ on } \Sigma, \\ \psi(T) = f^0, \quad \psi_t(T) = f^1. \end{cases} \tag{6.77}$$

With ψ_{ij}^n an approximation of $\psi(M_{ij}, n\Delta t)$, we approximate (6.77) by the following *explicit* FD scheme

$$\begin{cases} \dfrac{\psi_{ij}^{n-1} + \psi_{ij}^{n+1} - 2\psi_{ij}^n}{|\Delta t|^2} - \dfrac{\psi_{i+1j}^n + \psi_{i-1j}^n + \psi_{ij+1}^n + \psi_{ij-1}^n - 4\psi_{ij}^n}{h^2} = 0, \\ 1 \le i, j \le I, \quad 0 \le n \le N, \end{cases} \tag{6.78$_1$}$$

$$\psi_{kl}^n = 0 \quad \text{if } M_{kl} \in \Gamma, \tag{6.78$_2$}$$

$$\psi_{ij}^N = f^0(M_{ij}), \quad \psi_{ij}^{N+1} - \psi_{ij}^{N-1} = 2\Delta t f^1(M_{ij}), \quad 1 \le i, j \le I. \tag{6.78$_3$}$$

To be *stable*, the above scheme has to satisfy the following (*stability*) condition

$$\Delta t \le h/\sqrt{2}. \tag{6.79}$$

6.8.4.3. Approximation of $(\partial\psi/\partial n)|_\Sigma$. Suppose that we want to approximate $\partial\psi/\partial n$ at $M \in \Gamma$, as shown in Figure 44. Suppose that ψ is known at E; we shall then approximate $\partial\psi/\partial n$ at M by

$$\frac{\partial\psi}{\partial n}(M) \approx \frac{\psi(E) - \psi(W)}{2h}. \tag{6.80}$$

In fact, $\psi(E)$ is not known since $E \notin \bar{\Omega}$. However – formally at least – $\psi = 0$ on Σ implies $\psi_{tt} = 0$ on Σ, which combined with $\psi_{tt} - \Delta\psi = 0$ implies $\Delta\psi = 0$ on Σ; discretizing this last relation at M yields

$$\frac{\psi(W) + \psi(E) + \psi(N) + \psi(S) - 4\psi(M)}{h^2} = 0. \tag{6.81}$$

Since N, M, S belong to Γ, (6.81) reduces to

$$\psi(W) = -\psi(E), \tag{6.82}$$

which combined with (6.80) implies that

$$\frac{\partial\psi}{\partial n}(M) \approx -\frac{\psi(W)}{h} = \frac{0 - \psi(W)}{h} = \frac{\psi(M) - \psi(W)}{h}. \tag{6.83}$$

In that particular case, the *centred* approximation (6.80) (which is *second-order accurate*) coincides with the *one-sided* one in (6.83) (which is only *first-order accurate*, in general). In the sequel, we shall use, therefore, (6.83) to approximate $\partial\psi/\partial n$ at M and we shall denote by $\delta_{kl}\psi$ the corresponding approximation of $\partial\psi/\partial n$ at $M_{kl} \in \Gamma$.

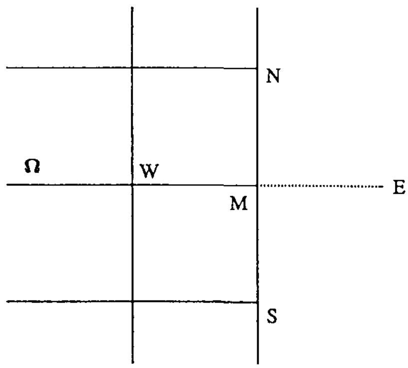


Fig. 44.

6.8.4.4. Approximation of the wave problem (6.33)$_2$. Similarly to (6.33)$_1$, the wave problem (6.33)$_2$, namely here,

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0 \text{ in } Q, \quad \varphi = \dfrac{\partial\psi}{\partial n} \text{ on } \Sigma, \\ \varphi(0) = 0, \quad \varphi_t(0) = 0 \end{cases} \tag{6.84}$$

will be approximated by

$$\begin{cases} \dfrac{\varphi_{ij}^{n+1} + \varphi_{ij}^{n-1} - 2\varphi_{ij}^n}{|\Delta t|^2} - \dfrac{\varphi_{i+1j}^n + \varphi_{i-1j}^n + \varphi_{ij+1}^n + \varphi_{ij-1}^n - 4\varphi_{ij}^n}{h^2} = 0, \\ 1 \le i, j \le I, \quad 0 \le n \le N, \end{cases} \tag{6.85$_1$}$$

$$\varphi_{kl}^n = \delta_{kl}\psi^n \quad \text{if } M_{kl} \in \Gamma, \tag{6.85$_2$}$$

$$\varphi_{ij}^0 = 0, \quad \frac{\varphi_{ij}^1 - \varphi_{ij}^{-1}}{2\Delta t} = 0, \quad 1 \le i, j \le I. \tag{6.85$_3$}$$

6.8.4.5. Approximation of Λ. Starting from

$$\mathbf{f}_h = \left\{\{f_{ij}^0, f_{ij}^1\}\right\}_{1 \le i,j \le I}$$

and via the solution of the discrete wave equations (6.78), (6.85) we approximate $\Lambda\mathbf{f}$ by

$$\Lambda_h^{\Delta t}\mathbf{f}_h = \left\{\left\{-\frac{\varphi_{ij}^{N+1} - \varphi_{ij}^{N-1}}{2\Delta t}, \varphi_{ij}^N\right\}\right\}_{1 \le i,j \le I} \tag{6.86}$$

It is proved in Glowinski *et al.* (1990, pp. 17–19) that we have (with obvious notation)

$$\begin{aligned}\langle \Lambda_h^{\Delta t}\mathbf{f}_h, \hat{\mathbf{f}}_h\rangle_{h,\Delta t} &= h^2 \sum_{1\leq i,j\leq I}\left[\varphi_{ij}^N \hat{f}_{ij}^1 - \left(\frac{\varphi_{ij}^{N+1}-\varphi_{ij}^{N-1}}{2\Delta t}\right)\hat{f}_{ij}^0\right] \\ &= h\Delta t \sum_{n=0}^{N} \alpha_n \sum_{M_{kl}\in\Gamma^*} \delta_{kl}\psi^n \delta_{kl}\hat{\psi}^n, \end{aligned} \tag{6.87}$$

where, in (6.87), $\alpha_0 = \alpha_N = \frac{1}{2}$, $\alpha_n = 1, \forall n = 1,\ldots,N-1$, and where $\Gamma^* = \Gamma$ minus the four corners $\{0,0\}$, $\{0,1\}$, $\{1,0\}$, $\{1,1\}$. It follows from (6.87) that $\Lambda_h^{\Delta t}$ is *symmetric* and *positive semi-definite*. Actually, it is proved in Glowinski *et al.* (1990, Section 6.2) that $\Lambda_h^{\Delta t}$ is *positive definite* if $T > T_{\min} \approx \Delta t/h$. This property implies that if $T(> 0)$ is given, it suffices to take $\Delta t/h$ sufficiently small to have exact boundary controllability for the discrete wave equation. This property is in contradiction with the continuous case where the exact boundary controllability property is lost if T is too small ($T < 1$ here). The reason for this discrepancy will be discussed in the following.

6.8.4.6. Approximation of the dual problem (6.50), (6.52). With $\mathbf{z}_h$ a convenient approximation of $\mathbf{z} = \{z^0, z^1\}$ we approximate problem (6.50), (6.52) by

$$\Lambda_h^{\Delta t}\mathbf{f}_h^{\Delta t} = \boldsymbol{\sigma}\mathbf{z}_h, \tag{6.88}$$

where, in (6.88), $\boldsymbol{\sigma}$ denotes the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In Glowinski *et al.* (1990, Section 6.3), one may find a discrete variation of the conjugate gradient algorithm (6.58)–(6.73) which can be used to solve the approximate problem (6.88).

6.8.5. Numerical solution of a test problem; ill-posedness of the discrete problem (6.88)

Following Glowinski *et al.* (1990, Section 7), Dean *et al.* (1989, Section 2.7), Glowinski (1992a, Section 2.7) we still consider the case $\Omega = (0,1)^2$, $\Gamma_0 = \Gamma$, $A = -\Delta$; we take $T = 3.75/\sqrt{2}$ (> 1, so that the exact controllability property holds) and $\mathbf{f} = \{f^0, f^1\}$ defined by

$$f^0(x_1, x_2) = \sin \pi x_1 \sin \pi x_2, \quad f^1 = -\pi\sqrt{2} f^0. \tag{6.89}$$

It is shown in Glowinski *et al.* (1990, Section 7) that using *separation of variables methods* we can compute a *Fourier Series* expansion of $\Lambda\mathbf{f}$. The corresponding functions z^0 and z^1 (both computed by *Fast Fourier Transform*) have been visualized on Figures 45 and 46, respectively (the graph on Figure 46 is the plot of $-z^1$).

From the above figures, z^0 is a *Lipschitz continuous function* which is not C^1; similarly, z^1 is *bounded* but *discontinuous*. On Figure 47, we have shown

Table 6. *Summary of numerical results (n.c.: no convergence)*

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
Number of conjugate gradient iterations	20	38	84	363	n.c.
$\\|f^0 - f^0_*\\|_{L^2(\Omega)}$	0.42×10^{-1}	0.18×10^{-1}	0.41×10^{-1}	3.89	n.c.
$\\|f^0 - f^0_*\\|_{H^1_0(\Omega)}$	0.65	0.54	2.54	498.1	n.c.
$\\|f^1 - f^1_*\\|_{L^2(\Omega)}$	0.20	0.64×10^{-1}	1.18	170.6	n.c.
$\\|u - u_*\\|_{L^2(\Sigma)}$	0.51	0.24	0.24	1.31	n.c.
$\\|u_*\\|_{L^2(\Sigma)}$	7.320	7.395	7.456	7.520	n.c.

the plot of the function $t \to \|\partial\psi/\partial n(t)\|_{L^2(\Gamma)}$ where ψ, given by

$$\psi(x,t) = \sqrt{2}\cos\pi\sqrt{2}\left(t - \frac{7}{2\sqrt{2}}\right)\sin\pi x_1 \sin\pi x_2,$$

is the solution of the wave equation (6.77) when f^0 and f^1 are given by (6.89); we recall that $\partial\psi/\partial n|_\Sigma (= u)$ is precisely the optimal Dirichlet control for which we have exact boundary controllability.

The numerical methods described in Sections 6.8.3 and 6.8.4 have been applied to the solution of the above test problem taking $\Delta t = h/\sqrt{2}$. Interestingly enough, the numerical results deteriorate as h and Δt converge to zero; moreover, taking Δt twice smaller, i.e. $\Delta t = h/2\sqrt{2}$, does not improve the situation. Also, the number of conjugate gradient iterations necessary to achieve convergence increases as h and Δt decrease. Results of the numerical experiments are reported on Table 6. In Table 6, f^0_*, f^1_* and u_*, are the computed values of f^0, f^1 and u respectively.

The most striking fact coming from Table 6 is the deterioration in the numerical results as h and Δt tend to zero; indeed, for $h = 1/128$, convergence was not achieved after 1000 iterations. To illustrate this deterioration further as h and $\Delta t \to 0$ we have compared, in Figures 48 to 51, f^0 and f^1 with their computed approximations f^0_* and f^1_*, for $h = 1/32$ and $1/64$; we observe that for $h = 1/64$ the variations in f^0_* and f^1_* are so large that we have been obliged to use a very large scale to be able to picture them (indeed we have plotted $-f^1$, $-f^1_*$)

If, for the same values of h, one takes Δt smaller than $h/\sqrt{2}$, the results remain practically the same. In Section 6.8.6, we shall try to analyse the reasons for this deterioration in the numerical results as $h \to 0$ and also to cure it. To conclude this section we observe that the error $\|u - u_*\|_{L^2(\Sigma)}$ deteriorates much more slowly as $h \to 0$ than the errors $f^0 - f^0_*$, $f^1 - f^1_*$; in fact, the approximate values $\|u_*\|_{L^2(\Sigma)}$ of $\|u\|_{L^2(\Sigma)}$ are quite good, even for

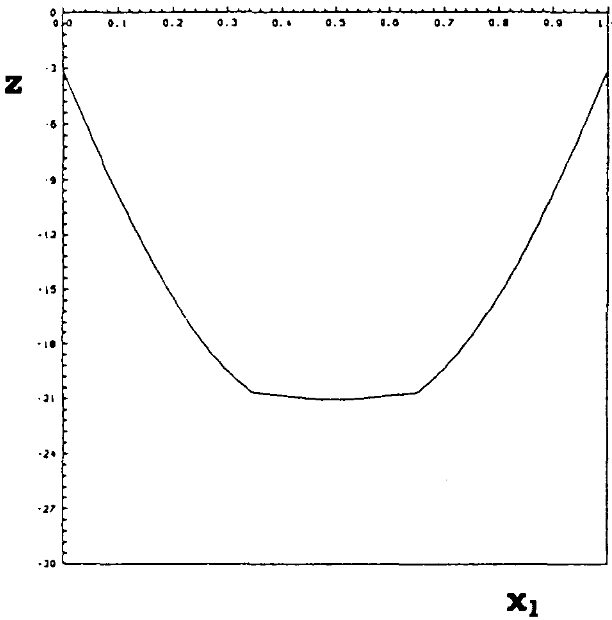


Fig. 45. $z^0(x_1, .5)$.

$h = 1/64$ if one realizes that the exact value of $\|u\|_{L^2(\Sigma)}$ is 7.386 68... For further illustrations and more details see Glowinski (1992a, Section 2.7) and the references therein.

6.8.6. Analysis and cures of the ill-posedness of the approximate problem (6.88)

It follows from the numerical results discussed in Section 6.8.5, that when h decreases to zero, the *ill-posedness* of the discrete problem gets worse. From the oscillatory results shown in Figures 48 to 51 it is quite clear that the trouble lies with the *high-frequency components* of the discrete solution or, to be more precise, with the way in which the discrete operator $\Lambda_h^{\Delta t}$ acts on the *short-wavelength* components of $\mathbf{f}_h$. Before analysing the mechanism producing these unwanted oscillations let us introduce a vector basis of $\mathbb{R}^{I \times I}$, well suited to the following discussion. This basis $\mathcal{B}_h$ is defined by

$$\mathcal{B}_h = \{w_{pq}\}_{1 \le p,q \le I}, \tag{6.90}$$

$$w_{pq} = \{\sin p\pi ih \times \sin q\pi jh\}_{1 \le i,j \le I}; \tag{6.91}$$

we recall that $h = 1/(I+1)$.

From the oscillatory results described in Section 6.8.5 it is reasonable to assume that the discrete operator $\Lambda_h^{\Delta t}$ damps too strongly those components of $\mathbf{f}_h^{\Delta t}$ with *large wavenumbers* p and q; in other words, we can expect that

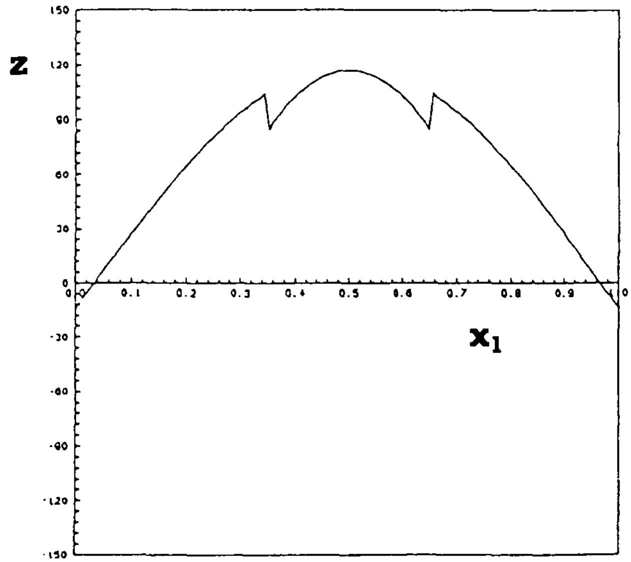


Fig. 46. $z^1(x_1, .5)$.

if p and/or q are large then $\Lambda_h^{\Delta t}\{w_{pq}, 0\}$ or $\Lambda_h^{\Delta t}\{0, w_{pq}\}$ will be quite small implying in turn (this is typical of ill-posed problems) that small perturbations of the right-hand side of the discrete problem (6.88) can produce very large variations in the corresponding solution.

Operator $\Lambda_h^{\Delta t}$ is fairly complicated (see Section 6.8.4 for its precise definition) and we can wonder which stage in it in particular acts as a *low pass filter* (i.e. selectively damping the large wavenumber components of the discrete solutions). Starting from the observation that the ill-posedness persists if, for a fixed h, we decrease Δt, it is then natural (and much simpler) to consider the *semi-discrete* case, where only the space derivatives have been discretized.

In such a case, problem (6.77) is discretized as follows (with $\dot{\psi} = \partial\psi/\partial t$, $\ddot{\psi} = \partial^2\psi/\partial t^2$) if $\Omega = (0,1)^2$ as in Sections 6.8.4, 6.8.5:

$$\ddot{\psi}_{ij} - \frac{\psi_{i+1j} + \psi_{i-1j} + \psi_{ij+1} + \psi_{ij-1} - 4\psi_{ij}}{h^2} = 0, \quad 1 \le i,j \le I, \qquad (6.92)_1$$

$$\psi_{kl} = 0 \text{ if } \{kh, lh\} \in \Gamma, \qquad (6.92)_2$$

$$\psi_{ij}(T) = f_h^0(ij, jh), \dot{\psi}_{ij}(T) = f_h^1(ih, jh), \quad 1 \le i,j \le I. \qquad (6.92)_3$$

Consider now the particular case where

$$f_h^0 = w_{pq}, \quad f_h^1 = 0. \qquad (6.93)$$

Since the vectors w_{pq} are for $1 \le p,q \le I$ the *eigenvectors* of the discrete

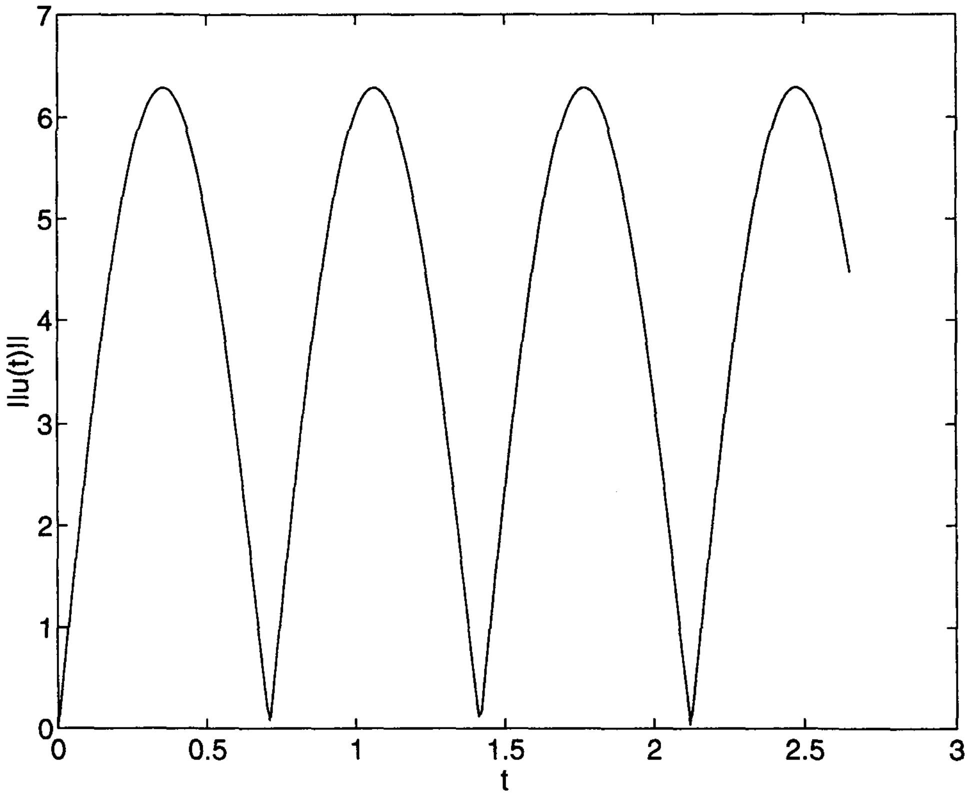


Fig. 47. $\|(\partial\psi/\partial n)(t)\|_{L^2(\Gamma)}$.

Laplace operator occurring in $(6.92)_1$ and that the corresponding eigenvalues $\lambda_{pq}(h)$ are given by

$$\lambda_{pq}(h) = \frac{4}{h^2}\left(\sin^2 p\pi\frac{h}{2} + \sin^2 q\pi\frac{h}{2}\right), \tag{6.94}$$

we can easily prove that the solution of (6.92), (6.93) is given by

$$\psi_{ij}(t) = \sin p\pi ih \sin q\pi jh \cos\left(\sqrt{\lambda_{pq}(h)}(T-t)\right), \quad 0 \le i,j \le I+1. \tag{6.95}$$

Next, we use (6.83) (see Section 6.8.4.3) to compute, from (6.95), the approximation of $\partial\psi/\partial n$ at the boundary point $M_{0j} = \{0, jh\}$, with $1 \le j \le I$; thus at time t, $\partial\varphi/\partial n$ is approximated at M_{0j} by

$$\delta\psi_h(M_{0j}, t) = -\frac{1}{h}\sin p\pi h \sin q\pi jh \cos\left(\sqrt{\lambda_{pq}(h)}(T-t)\right). \tag{6.96}$$

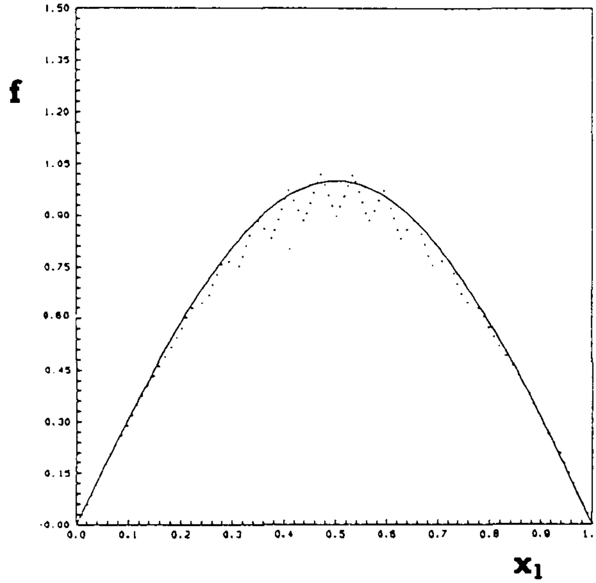


Fig. 48. Variations of $f^0(x_1,.5)$ (——) and $f^0_*(x_1,.5)$ $(\cdots\cdots)$ $(h=1/32)$.

If $1 \le p \ll I$, the coefficient $K_h(p)$ defined by

$$K_h(p) = \frac{\sin p\pi h}{h} \tag{6.97}$$

is an approximation of $p\pi$ which is second-order accurate (with respect to h); now if $p \sim I/2$ we have $K_h(p) \sim I$ and if $p = I$ we have (since $h = 1/(I+1))K_h(I) \sim \pi$.

Back to the *continuous problem*, it is quite clear that (6.92), (6.93) is in fact a semi-discrete approximation of the wave problem

$$\psi_{tt} - \Delta\psi = 0 \text{ in } Q, \quad \psi = 0 \text{ on } \Sigma, \qquad (6.98)_1$$

$$\psi(x,T) = \sin p\pi x_1 \sin q\pi x_2, \quad \psi_t(x,T) = 0. \qquad (6.98)_2$$

The solution of (6.98) is given by

$$\psi(x,t) = \sin p\pi x_1 \sin q\pi x_2 \cos\left(\pi\sqrt{p^2+q^2}(T-t)\right). \tag{6.99}$$

Computing $(\partial\psi/\partial n)|_\Sigma$ we obtain

$$\frac{\partial\psi}{\partial n}(M_{0j},t) = -p\pi \sin q\pi jh \cos\left(\pi\sqrt{p^2+q^2}(T-t)\right). \tag{6.100}$$

We observe that if $p \ll I$ and $q \ll I$, then $(\partial\psi/\partial n)(M_{0j},t)$ and $\delta\psi_h(M_{0j},t)$ are close quantities. Now, if the wavenumber p is large, then the coefficient $K(p) = \pi p$ in (6.100) is much larger than the corresponding coefficient $K_h(p)$

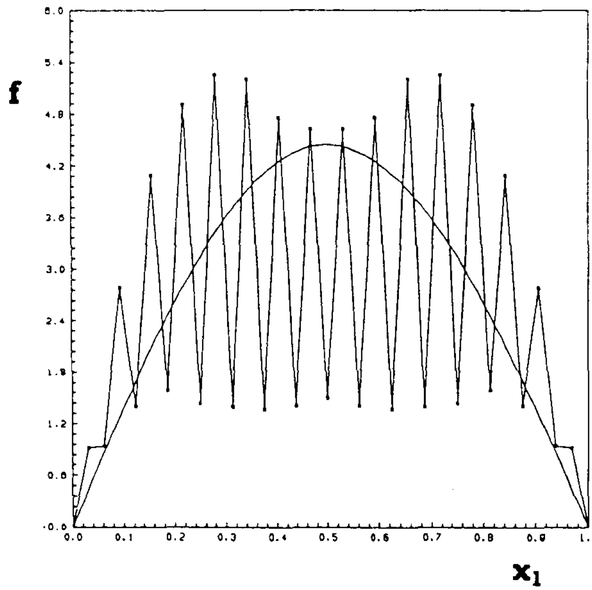


Fig. 49. Variations of $f^1(x_1,.5)$ (——) and $f_*^1(x_1,.5)$ $(\cdots\cdots)$ $(h=1/32)$.

in (6.97); we have, in fact,

$$\frac{K(I/2)}{K_h(I/2)} \approx \frac{\pi}{2}, \quad \frac{K(I)}{K_h(I)} \approx I.$$

Figure 52, (where we have visualized, with an appropriate scaling, the function $p\pi \to p\pi$ and its discrete analogue, namely the function $p\pi \to \sin p\pi h/h$) shows that for $p,q > (I+1)/2$, the approximate normal derivative operator introduces a very strong damping. We would have obtained similar results by considering, instead of (6.93), initial conditions such as

$$f_h^0 = 0, \quad f_h^1 = w_{pq}. \tag{6.101}$$

From the above analysis it appears that the approximation of $(\partial\psi/\partial n)|_\Sigma$, which is used to construct operator $\Lambda_h^{\Delta t}$, introduces very strong damping of the *large wavenumber components* of $\mathbf{f}_h$. Possible cures for the ill-posedness of the discrete problem have been discussed in Glowinski *et al.* (1990), Dean *et al.* (1989), Glowinski (1992a). The first reference, in particular, contains a detailed discussion of a *biharmonic Tychonoff regularization procedure*, where problem (6.50) is approximated by a discrete version of

$$\varepsilon\mathbf{M}\mathbf{f}_\varepsilon + \Lambda\mathbf{f}_\varepsilon = \begin{pmatrix} -z^1 \\ z^0 \end{pmatrix} \text{ in } \Omega, \tag{6.102$_1$}$$

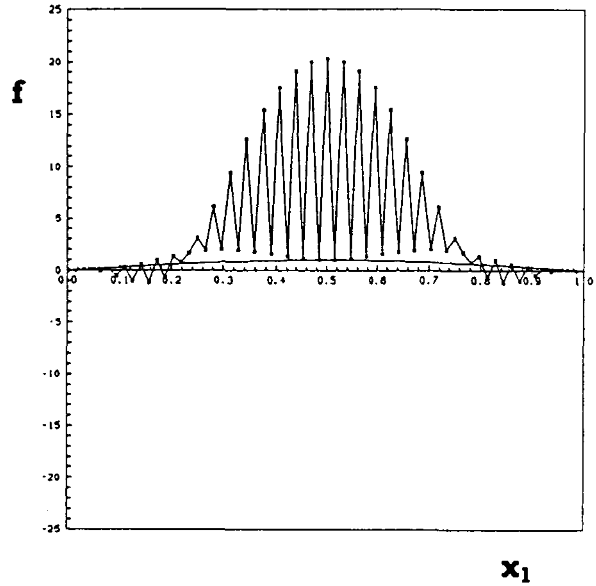


Fig. 50. Variations of $f^0(x_1, .5)$ (——) and $f^0_*(x_1, .5)$ (· · · · · ·) ($h = 1/64$).

$$\Delta f^0_\varepsilon = f^0_\varepsilon = f^1_\varepsilon = 0 \text{ on } \Gamma,$$

where, in (6.102), $\varepsilon > 0$, $\mathbf{f}_\varepsilon = \{f^0_\varepsilon, f^1_\varepsilon\}$, and where operator $\mathbf{M}$ is defined by

$$\mathbf{M} = \begin{pmatrix} \Delta^2 & 0 \\ 0 & -\Delta \end{pmatrix}. \qquad (6.102)_2$$

Various theoretical and numerical issues associated with (6.102) are discussed in Glowinski *et al.* (1990), including the choice of ε as a function of h; indeed elementary *boundary layer* considerations show that ε has to be of the order of h^2. The numerical results presented in Glowinski *et al.* (1990) and Dean *et al.* (1989) validate convincingly the above regularization approach. Also in Glowinski *et al.* (1990, p. 42) we suggest that *mixed* FE approximations (see, e.g. Roberts and Thomas (1991), Brezzi and Fortin (1991) for introductions to mixed FE methods) may improve the quality of the numerical results; one of the reasons for this potential improvement is that mixed FE methods are known to provide accurate approximations of derivatives and also that derivative values at selected nodes (including boundary ones) are natural degrees of freedom for these approximations. As shown in Glowinski, Kinton and Wheeler (1989) and Dupont, Glowinski, Kinton and Wheeler (1992) this approach substantially reduces the unwanted oscillations, since *without* any regularization good numerical results have been obtained using mixed FE implementation of HUM. The main drawback of

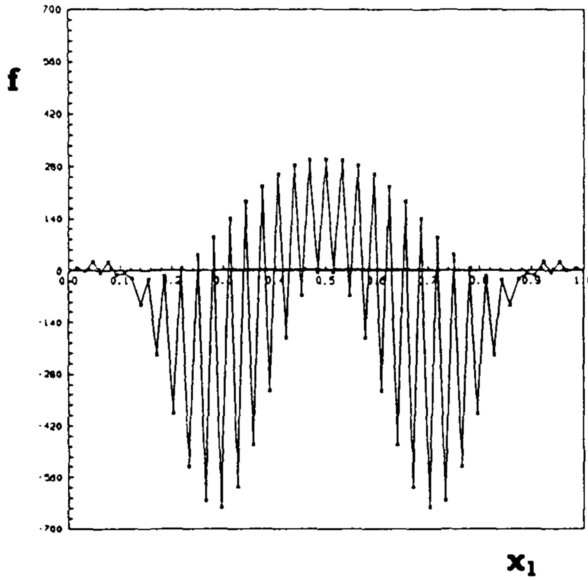


Fig. 51. Variations of $f^1(x_1, .5)$ (——) and $f^1_*(x_1, .5)$ (······) ($h = 1/64$).

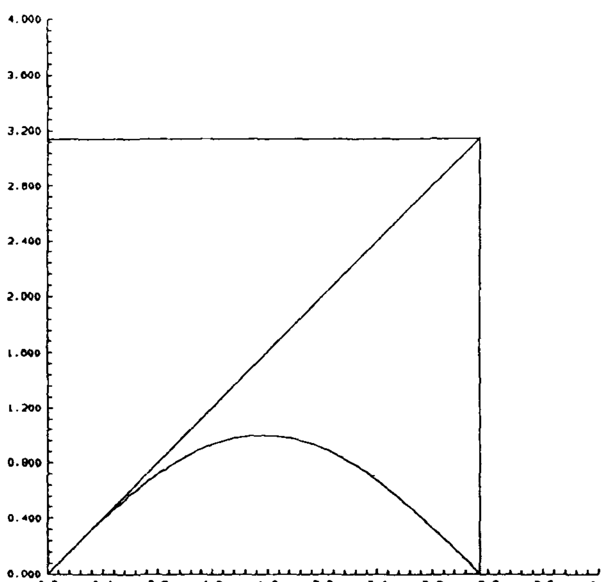


Fig. 52.

the mixed FE approach is that (without regularization) the number of conjugate gradient iterations necessary to achieve convergence increases (slowly) with h (in fact, roughly, as $h^{-1/2}$); it seems, also, on the basis of numerical

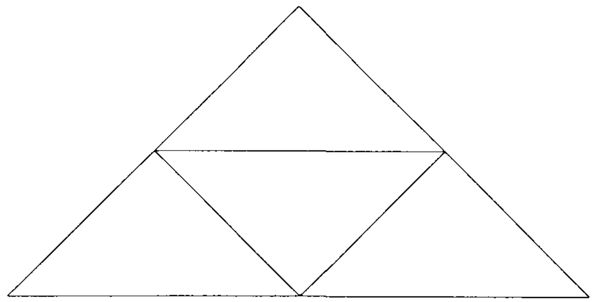


Fig. 53. Triangles of $\mathcal{T}_h$ and $\mathcal{T}_{h/2}$.

experiments, that the level of unwanted oscillations increases (slowly, again) with T.

Another cure for spurious oscillations has been introduced in Glowinski and Li (1990) (see also Glowkinski (1992a, Section 3)); this (simple) cure, suggested by Figure 52, consists of eliminating the *short-wavelength* components of $\mathbf{f}_h$ with wavenumbers p and q larger than $(I+1)/2$; to achieve this radical filtering it suffices to define $\mathbf{f}_h$ on an FD grid of step size $\geq 2h$. An FE implementation of the above filtering technique is discussed in Section 6.8.7; also, for the calculations described in Section 6.9 we have defined $\mathbf{f}_h$ over a grid of step size $2h$.

6.8.7. An FE implementation of the filtering technique of Section 6.8.6

6.8.7.1. Generalities. We go back to the case where (possibly) $\Omega \neq (0,1)^2$, $\Gamma_0 \neq \Gamma$ and $A \neq -\Delta$; the most natural fashion of combining HUM and the filtering technique discussed in Section 6.8.6 is to use *finite elements* for the space approximation; in fact, as shown in Glowinski *et al.* (1990, Section 6.2), special triangulations (like the one shown in Figure 1 of Section 2.6.1) will give back FD approximations closely related to the one discussed in Section 6.8.6. For simplicity, we suppose that Ω is a polygonal domain of $\mathbb{R}^2$; we then introduce a triangulation $\mathcal{T}_h$ of Ω such that $\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} T$, with h the length of the largest edge (s) of $\mathcal{T}_h$. From $\mathcal{T}_h$, we define $\mathcal{T}_{h/2}$ by joining (see Figure 53), the midpoints of the edges of the triangles of $\mathcal{T}_h$.

With P_1 the space of the polynomials in two variables of degree ≤ 1, we define the spaces H_h^1 and H_{0h}^1 by

$$H_h^1 = \{v \mid \in C^0(\bar{\Omega}), v|_T \in P_1, \forall T \in \mathcal{T}_h\}, \quad H_{0h}^1 = \{v \mid v \in H_h^1, v|_\Gamma = 0\}; \tag{6.103}$$

similarly, we define $H_{h/2}^1$ and $H_{0h/2}^1$ by replacing h by $h/2$ in (6.103). We observe that $H_h^1 \subset H_{h/2}^1$, $H_{0h}^1 \subset H_{0h/2}^1$. We then approximate the $L^2(\Omega)$-

scalar product over H_h^1 by

$$(v, w)_h = \frac{1}{3}\sum_Q \omega_Q v(Q)w(Q), \ \forall v, w \in H_h^1, \tag{6.104}$$

where, in (6.104), Q describes the set of the vertices of $\mathcal{T}_h$ and where ω_Q is the area of the polygonal domain, union of those triangles of $\mathcal{T}_h$, with Q as a common vertex. Similarly, we define $(\cdot,\cdot)_{h/2}$ by substituting $h/2$ to h in (6.104).

Finally, assuming that the points at the interface of Γ_0 and $\Gamma\backslash\Gamma_0$ are vertices of $\mathcal{T}_{h/2}$, we define $V_{0h/2}$ by

$$V_{0h/2} = \{v \mid v \in H_{h/2}^1, \ v = 0 \text{ on } \Gamma_0\backslash\Gamma\}. \tag{6.105}$$

6.8.7.2. Approximation of problem (6.50). We approximate the fundamental equation $\Lambda f = \{-z^1, z^0\}$ by the following linear variational problem in $H_{0h}^1 \times H_{0h}^1$:

$$\begin{cases} \mathbf{f}_h^{\Delta t} \in H_{0h}^1 \times H_{0h}^1, \\ \lambda_h^{\Delta t}(\mathbf{f}_h, \mathbf{v}) = -\langle z^1, v^0\rangle + \int_\Omega z^0 v^1 \,\mathrm{d}x, \ \forall \mathbf{v} = \{v^0, v^1\} \in H_{0h}^1 \times H_{0h}^1. \end{cases} \tag{6.106}$$

In (6.106), $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, and the bilinear form $\lambda_h^{\Delta t}(\cdot,\cdot)$ is defined as follows.

(i) Take $\hat{\mathbf{f}}_h = \{\hat{f}_h^0, \hat{f}_h^1\} \in H_{0h}^1 \times H_{0h}^1$ and solve, for $n = N, \ldots, 0$, the *discrete variational problem*

$$\begin{cases} \hat{\psi}_h^{n-1} \in H_{0h/2}^1, \\ (\hat{\psi}_h^{n-1} + \hat{\psi}_h^{n+1} - 2\hat{\psi}_h^n, v)_{h/2} + |\Delta t|^2 a(\hat{\psi}_h^n, v) = 0, \ \forall v \in H_{0h/2}^1, \end{cases} \tag{6.107}$$

with the final conditions

$$\hat{\psi}_h^N = \hat{f}_h^0, \quad \hat{\psi}_h^{N+1} - \hat{\psi}_h^{N-1} = 2\Delta t \hat{f}_h^1; \tag{6.108}$$

we recall that $a(\cdot,\cdot)$ denotes the bilinear form defined by

$$a(v, w) = \langle Av, w\rangle, \ \forall v \in H^1(\Omega), \quad w \in H_0^1(\Omega).$$

(ii) To approximate $\partial\psi/\partial n_A$ over Σ_0, first introduce the complementary subspace $M_{h/2}$ of $H_{0h/2}^1$ defined by

$$\begin{cases} M_{h/2} \oplus H_{0h/2}^1 = V_{0h/2}, \\ v \in M_{h/2} \Longrightarrow v|_T = 0, \ \forall T \in \mathcal{T}_{h/2} \text{ such that } T \cap \Gamma = \emptyset; \end{cases} \tag{6.109}$$

we observe that $M_{h/2}$ is isomorphic to the space $\gamma V_{0h/2}$ of the traces over Γ_0 of the functions of $V_{0h/2}$. The approximation of $(\partial\hat{\psi}/\partial n_A)|_{\Gamma_0}$ at $t = n\Delta t$ is then defined (cf. Glowinski *et al.* (1990) and Section 2.4.3) by solving the linear variational problem

$$\begin{cases} \delta\hat{\psi}_h^n \in \gamma V_{0h/2}, \\ \int_{\Gamma_0} \delta\hat{\psi}_h^n v \, d\Gamma = a(\hat{\psi}_h^n, v), \ \forall v \in M_{h/2}. \end{cases} \tag{6.110}$$

Variants of (6.110), leading to linear systems with diagonal matrices are given in Glowinski *et al.* (1990).

(iii) Now, for $n = 0, \ldots, N$; solve the *discrete variational problem*

$$\begin{cases} \hat{\varphi}_h^{n+1} \in V_{0h/2}; \hat{\varphi}_h^{n+1} = \delta\hat{\psi}_h^{n+1} \text{ on } \Gamma_0, \\ (\hat{\varphi}_h^{n+1} + \hat{\varphi}_h^{n-1} - 2\hat{\varphi}_h^n, v)_{h/2} + |\Delta t|^2 a(\hat{\varphi}_h^n, v) = 0, \ \forall v \in H^1_{0h/2} \end{cases} \tag{6.111}$$

initialized via

$$\hat{\varphi}_h^0 = 0, \quad \hat{\varphi}_h^1 - \hat{\varphi}_h^{-1} = 0. \tag{6.112}$$

(iv) Finally, define $\lambda_h^{\Delta t}(\cdot,\cdot)$ by

$$\lambda_h^{\Delta t}(\hat{\mathbf{f}}_h, \mathbf{v}) = (\hat{\lambda}_h^0, v^0)_{h/2} + (\hat{\lambda}_h^1, v^1)_{h/2}, \ \forall \mathbf{v} = \{v^0, v^1\} \in H^1_{0h} \times H^1_{0h}, \tag{6.113}$$

where, in (6.113), $\hat{\lambda}_h^0$ and $\hat{\lambda}_h^1$ both belong to $H^1_{0h/2}$ and satisfy

$$\begin{aligned} \hat{\lambda}_h^0(P) &= -\frac{\hat{\varphi}_h^{N+1}(P) - \hat{\varphi}_h^{N-1}(P)}{2\Delta t}, \\ \hat{\lambda}_h^1(P) &= \hat{\varphi}_h^N(P), \ \forall P \text{ interior vertex of } \mathcal{T}_{h/2}. \end{aligned} \tag{6.114}$$

Following Glowinski *et al.* (1990, Section 6) we can prove that

$$\lambda_h^{\Delta t}(\mathbf{f}_{1h}, \mathbf{f}_{2h}) = \Delta t \sum_{n=0}^{N} \alpha_n \int_{\Gamma_0} \delta\psi_{1h}^n \delta\psi_{2h}^n \, d\Gamma, \ \forall f_{1h}, f_{2h} \in H^1_{0h} \times H^1_{0h}, \tag{6.115}$$

where, in (6.115), $\alpha_0 = \alpha_N = 1/2$, and $\alpha_n = 1$ if $0 < n < N$.

It follows from (6.115) that the bilinear form $\lambda_h^{\Delta t}(\cdot,\cdot)$ is *symmetric* and *positive semi-definite*. As in Glowinski *et al.* (1990, Section 6.2), we should prove that $\lambda_h^{\Delta t}(\cdot,\cdot)$ is *positive definite* if T is sufficiently large and if Ω is a square (or a rectangle) and $\mathcal{T}_h$, $\mathcal{T}_{h/2}$ *regular* triangulations of Ω. From the properties of $\lambda_h^{\Delta t}(\cdot,\cdot)$ the linear variational problem (6.106) (which approximates problem (6.50)) can be solved by a *conjugate gradient algorithm* operating in $H^1_{0h} \times H^1_{0h}$. This algorithm is described in Section 6.8.7.3.

6.8.7.3. Conjugate gradient solution of the approximate problem (6.106). The conjugate gradient algorithm for solving problem (6.106) is an FE implementation of algorithm (6.58)–(6.73) (see Section 6.8.3).

Description of the Conjugate Gradient Algorithm

Step 0: Initialization

$$f_0^0 \in H^1_{0h}, \quad f_0^1 \in H^1_{0h} \text{ are given;} \tag{6.116}$$

solve then, for $n = N, N-1, \dots, 0$, the discrete linear variational problem

$$\begin{cases} \psi_0^{n-1} \in H^1_{0h/2}, \\ \left(\dfrac{\psi_0^{n-1} + \psi_0^{n+1} - 2\psi_0^n}{|\Delta t|^2}, v\right)_{h/2} + a(\psi_0^n, v) = 0, \ \forall v \in H^1_{0h/2}, \end{cases} \tag{6.117}$$

initialized by

$$\psi_0^N = f_0^0, \quad \psi_0^{N+1} - \psi_0^{N-1} = 2\Delta t f_0^1, \tag{6.118}$$

and store ψ_0^0, ψ_0^{-1}.

Then for $n = 0, 1, \dots, N$, compute ψ_0^n, $\delta\psi_0^n$, φ_0^{n+1} by forward (discrete) time integration, as follows.

1 *If $n = 0$, compute $\delta\psi_0^0$ from ψ_0^0 using (6.110).*
If $n > 0$, compute first ψ_0^n by solving

$$\begin{cases} \psi_0^n \in H^1_{0h/2}, \\ \left(\dfrac{\psi_0^n + \psi_0^{n-2} - 2\psi_0^{n-1}}{|\Delta t|^2}, v\right)_{h/2} + a(\psi_0^{n-1}, v) = 0, \ \forall v \in H^1_{0h/2} \end{cases} \tag{6.119}$$

and then $\delta\psi_0^n$ by using (6.110).

2 *Take $\varphi_0^n = \delta\psi_0^n$ on Γ_0 and use*

$$\left(\frac{\varphi_0^{n+1} + \varphi_0^{n-1} - 2\varphi_0^n}{|\Delta t|^2}, v\right)_{h/2} + a(\varphi_0^n, v) = 0, \ \forall v \in H^1_{0h/2}, \tag{6.120}$$

to compute the values taken by $\varphi_0^{n+1} (\in V_{0h/2})$ at the interior vertices of $\mathcal{T}_{h/2}$. These calculations are initialized by

$$\varphi_0^0(P) = 0, \varphi_0^1(P) - \varphi_0^{-1}(P) = 0, \ \forall P \text{ interior vertex of } \mathcal{T}_{h/2}. \tag{6.121}$$

Compute then $\mathbf{g}_0 = \{g_0^0, g_0^1\} \in H^1_{0h} \times H^1_{0h}$ by solving the following discrete Dirichlet problem

$$\begin{cases} g_0^0 \in H^1_{0h}, \\ \displaystyle\int_\Omega \nabla g_0^0 \cdot \nabla v \,\mathrm{d}x = \langle z^1, v\rangle - \left(\frac{\varphi_0^{N+1} - \varphi_0^{N-1}}{2\Delta t}, v\right)_{h/2}, \ \forall v \in H^1_{0h}, \end{cases} \tag{6.122}$$

and then

$$\begin{cases} g_0^1 \in H^1_{0h}, \\ (g_0^1, v)_h = (\varphi_0^N, v)_{h/2} - \displaystyle\int_\Omega z^0 v \,\mathrm{d}x, \ \forall v \in H^1_{0h}. \end{cases} \tag{6.123}$$

If $\mathbf{g}_0 = \mathbf{0}$, or is 'small', take $\mathbf{f}_h^{\Delta t} = \mathbf{f}_0$; if not, set

$$\mathbf{w}_0 = \mathbf{g}_0. \tag{6.124}$$

Then for $k \geq 0$, assuming that $\mathbf{f}_k$, $\mathbf{g}_k$, $\mathbf{w}_k$ are known, compute $\mathbf{f}_{k+1}$, $\mathbf{g}_{k+1}$, $\mathbf{w}_{k+1}$ as follows.

Step 1: Descent
For $n = N, N-1, \ldots, 0$, solve the discrete backward wave equation

$$\begin{cases} \bar{\psi}_k^{n-1} \in H^1_{0h/2}, \\ \left(\dfrac{\bar{\psi}_k^{n-1} + \bar{\psi}_k^{n+1} - 2\bar{\psi}_k^n}{|\Delta t|^2}, v\right)_{h/2} + a(\bar{\psi}_k^n, v) = 0, \ \forall v \in H^1_{0h/2}, \end{cases} \tag{6.125}$$

initialized by

$$\bar{\psi}_k^N = w_k^0, \quad \bar{\psi}_k^{N+1} - \bar{\psi}_k^{N-1} = 2\Delta t w_k^1, \tag{6.126}$$

and store $\bar{\psi}_k^0$, $\bar{\psi}_k^{-1}$.
Then for $n = 0, 1, \ldots, N$, compute $\bar{\psi}_k^n$, $\delta\bar{\psi}_k^n$, $\bar{\varphi}_k^{n+1}$ by forward time integration as follows.

1 *If $n = 0$, compute $\delta\bar{\psi}_k^0$ from $\bar{\psi}_k^0$ using (6.110).*
If $n > 0$, compute first $\bar{\psi}_k^n$ by solving

$$\begin{cases} \bar{\psi}_k^n \in H^1_{0h/2}, \\ \left(\dfrac{\bar{\psi}_k^n + \bar{\psi}_k^{n-2} - 2\bar{\psi}_k^{n-1}}{|\Delta t|^2}, v\right)_{h/2} + a(\bar{\psi}_k^{n-1}, v) = 0, \ \forall v \in H^1_{0h/2}, \end{cases} \tag{6.127}$$

and then $\delta\bar{\psi}_k^n$ by using (6.110).

2 *Take $\bar{\varphi}_k^n = \delta\bar{\psi}_k^n$ on Γ_0 and use*

$$\left(\frac{\bar{\varphi}_k^{n+1} + \bar{\varphi}_k^{n-1} - 2\bar{\varphi}_k^n}{|\Delta t|^2}, v\right)_{h/2} + a(\bar{\varphi}_k^n, v) = 0, \ \forall v \in H^1_{0h/2}, \tag{6.128}$$

to compute the values taken by $\bar{\varphi}_k^{n+1} (\in V_{0h/2})$ at the interior vertices of $\mathcal{T}_{h/2}$. These calculations are initialized by

$$\bar{\varphi}_k^1(P) - \bar{\varphi}_k^{-1}(P) = \bar{\varphi}_k^0(P) = 0, \ \forall P \text{ interior vertex of } \mathcal{T}_{h/2}. \tag{6.129}$$

Compute now $\mathbf{g}_k (= \{g_k^0, g_k^1\}) \in H^1_{0h} \times H^1_{0h}$ by

$$\begin{cases} \bar{g}_k^0 \in H^1_{0h}, \\ \displaystyle\int_\Omega \nabla \bar{g}_k^0 \cdot \nabla v \, dx = -\left(\frac{\bar{\varphi}_k^{N+1} - \bar{\varphi}_k^{N-1}}{2\Delta t}, v\right)_{h/2}, \ \forall v \in H^1_{0h}, \end{cases} \tag{6.130}$$

$$\begin{cases} \bar{g}_k^1 \in H^1_{0h}, \\ (\bar{g}_k^1, v)_h = (\bar{\varphi}_k^N, v)_{h/2}, \ \forall v \in V_{0h}, \end{cases} \tag{6.131}$$

and then ρ_k *by*

$$\rho_k = \int_\Omega |\nabla g_k^0|^2 \, dx + (g_k^1, g_k^1)_h \Big/ \int_\Omega \nabla \bar{g}_k^0 \cdot \nabla w_k^0 \, dx + (\bar{g}_k^1, w_k^1)_h. \quad (6.132)$$

Once ρ_k *is known, compute*

$$\mathbf{f}_{k+1} = \mathbf{f}_k - \rho_k \mathbf{w}_k, \quad (6.133)$$

$$\mathbf{g}_{k+1} = \mathbf{g}_k - \rho_k \bar{\mathbf{g}}_k. \quad (6.134)$$

Step 2. Test of the convergence and construction of the new descent direction

If $\mathbf{g}_{k+1} = \mathbf{0}$, *or is 'small', take* $\mathbf{f}_h^{\Delta t} = \mathbf{f}_{k+1}$; *if not, compute*

$$\gamma_k = \int_\Omega |\nabla g_{k+1}^0|^2 \, dx + (g_{k+1}^1, g_{k+1}^1)_h \Big/ \int_\Omega |g_k^0|^2 \, dx + (g_k^1, g_k^1)_h, \quad (6.135)$$

and set

$$\mathbf{w}_{k+1} = \mathbf{g}_{k+1} + \gamma_k \mathbf{w}_k. \quad (6.136)$$

Do $k = k + 1$ *and go to* (6.125).

Remark 6.18 The above algorithm may seem a little bit complicated at first glance (21 statements); in fact, it is fairly easy to implement, since the only nontrivial part of it is the solution (on the coarse grid) of the discrete Dirichlet problems (6.122) and (6.130). An interesting feature of algorithm (6.116)–(6.136) is that the *forward integration* of the discrete wave equations (6.117) and (6.125) provides a very substantial computer memory saving. To illustrate this claim, let us consider the case where $\Omega = (0,1) \times (0,1)$, $\Gamma_0 = \Gamma$, $T = 2\sqrt{2}$, $h = 1/64$, $\Delta t = h/2\sqrt{2} = \sqrt{2}/256$; we have then – approximately – $(512)^2$ discretization points on Σ, therefore in that specific case, using algorithm (6.116)–(6.136) avoids the storage of 2.62×10^5 real numbers. The saving would be even more substantial for larger T and would be an absolute necessity for three-dimensional problems. In fact, the above storage-saving strategy which is based on the *time reversibility* of the *wave equation* (6.1) cannot be applied to the control problems discussed in Sections 1 and 2 since they concern systems modelled by *diffusion* equations which are, unfortunately, *time irreversible*.

Remark 6.19 The above remark shows the interest of solving the *dual problem* from a computational point of view. In the original control problem, the unknown is the control u which is defined over Σ_0; for the dual problem the unknown is then the solution $\mathbf{f}$ of problem (6.50). If one considers again the particular case of Remark 6.18, i.e. $\Omega = (0,1) \times (0,1)$, $\Gamma_0 = \Gamma$, $T = 2\sqrt{2}$, $h = 1/64$, $\Delta t = h/2\sqrt{2}$ the unknown u will be approximated by a finite dimensional vector $u_h^{\Delta t}$ with 2.62×10^5 components, while $\mathbf{f}$ is approximated by $\mathbf{f}_h^{\Delta t}$ of dimension $2 \times (63)^2 = 7.938 \times 10^3$, a substantial saving indeed. Also, the dimension of $\mathbf{f}_h^{\Delta t}$ remains the same as T increases, while the dimension of $u_h^{\Delta t}$ is proportional to T.

Numerical results obtained using algorithm (6.116)–(6.136) will be discussed in Section 6.9.

6.8.8. Solution of the approximate boundary controllability problem (6.24)

Following the approach advocated for the exact boundary controllability problem, we shall address the *numerical solution* of the *approximate boundary controllability* problem (6.24) via the solution of its *dual problem*, namely problem (6.37), (6.38). This can also be formulated as

$$\Lambda \mathbf{f} + \partial j(\mathbf{f}) = \begin{pmatrix} -z^1 \\ z^0 \end{pmatrix}, \tag{6.137}$$

where, in (6.137), the *convex functional* $j : H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ is defined by

$$j(\hat{\mathbf{f}}) = \beta_1 \|\hat{f}^0\|_{H_0^1(\Omega)} + \beta_0 \|\hat{f}^1\|_{L^2(\Omega)}, \ \forall \hat{\mathbf{f}} = \{\hat{f}^0, \hat{f}^1\} \in H_0^1(\Omega) \times L^2(\Omega). \tag{6.138}$$

Following a strategy already used in preceding sections (see, e.g. Section 1.8.8) we associate with the 'elliptic' problem (6.137) the following *initial value problem*

$$\begin{cases} \begin{pmatrix} -\Delta & 0 \\ 0 & I \end{pmatrix} \dfrac{\partial \mathbf{f}}{\partial \tau} + \Lambda \mathbf{f} + \partial j(\mathbf{f}) = \begin{pmatrix} -z^1 \\ z^0 \end{pmatrix}, \\ \mathbf{f}(0) = \mathbf{f}_0, \end{cases} \tag{6.139}$$

where τ is a pseudo-time. The particular form of problem (6.139) clearly suggests time integration by *operator splitting* (see, again, Section 1.8.8). Concentrating on the *Peaceman–Rachford scheme*, we obtain – with $\Delta\tau (> 0)$ a pseudo-time step – the following algorithm to compute the steady-state solution of problem (6.139), i.e. the solution of problem (6.37), (6.38), (6.137):

$$\mathbf{f}^0 = \mathbf{f}_0; \tag{6.140}$$

then, for $k \geq 0$, assuming that $\mathbf{f}^k$ is known, we compute $\mathbf{f}^{k+1/2}$ and $\mathbf{f}^{k+1}$ via the solution of

$$\begin{pmatrix} -\Delta & 0 \\ 0 & I \end{pmatrix} \frac{\mathbf{f}^{k+1/2} - \mathbf{f}^k}{\Delta\tau/2} + \Lambda \mathbf{f}^k + \partial j(\mathbf{f}^{k+1/2}) = \begin{pmatrix} -z^1 \\ z^0 \end{pmatrix}, \tag{6.141}$$

$$\begin{pmatrix} -\Delta & 0 \\ 0 & I \end{pmatrix} \frac{\mathbf{f}^{k+1} - \mathbf{f}^{k+1/2}}{\Delta\tau/2} + \Lambda \mathbf{f}^{k+1} + \partial j(\mathbf{f}^{k+1/2}) = \begin{pmatrix} -z^1 \\ z^0 \end{pmatrix}. \tag{6.142}$$

Let us discuss the solution of the subproblems (6.141), (6.142):

(i) Assuming that (6.141) has been solved, equation (6.142) can be formulated as

$$\begin{pmatrix} -\Delta & 0 \\ 0 & I \end{pmatrix} \frac{\mathbf{f}^{k+1} - 2\mathbf{f}^{k+1/2} + \mathbf{f}^k}{\Delta\tau/2} + \Lambda\mathbf{f}^{k+1} = \Lambda\mathbf{f}^k,$$

i.e.

$$\begin{pmatrix} -\Delta & 0 \\ 0 & I \end{pmatrix} \mathbf{f}^{k+1} + \frac{\Delta\tau}{2}\Lambda\mathbf{f}^{k+1} = \begin{pmatrix} -\Delta & 0 \\ 0 & I \end{pmatrix} (2\mathbf{f}^{k+1/2} - \mathbf{f}^k) + \frac{\Delta\tau}{2}\Lambda\mathbf{f}^k. \quad (6.143)$$

Problem (6.143) is a variant of problem (6.50) (a regularized one, in fact) and can be solved by a conjugate gradient algorithm closely related to algorithm (6.58)–(6.73) (we have to replace the bilinear form

$$\{\mathbf{f}_1, \mathbf{f}_2\} \to \langle \Lambda\mathbf{f}_1, \mathbf{f}_2 \rangle : (H_0^1(\Omega) \times L^2(\Omega))^2 \to \mathbb{R}$$

by

$$\{\mathbf{f}_1, \mathbf{f}_2\} \to \int_\Omega \nabla f_1^0 \cdot \nabla f_2^0 \, dx + \int_\Omega f_1^1 f_2^1 \, dx + \frac{1}{2}\Delta\tau\langle \Lambda\mathbf{f}_1, \mathbf{f}_2\rangle)$$

(ii) Concerning the solution of problem (6.141), we shall take advantage of the fact that operator $\partial j(\cdot)$ is *diagonal* from $H_0^1(\Omega) \times L^2(\Omega)$ into $H^{-1}(\Omega) \times L^2(\Omega)$; solving problem (6.141) is then equivalent to solving the two following *uncoupled* minimization problems (where the notation is fairly obvious):

$$\min_{\hat{f}^0 \in H_0^1(\Omega)} \Bigg[\frac{1}{2} \int_\Omega |\nabla \hat{f}^0|^2 \, dx + \beta_1 \frac{\Delta\tau}{2} \left(\int_\Omega |\nabla \hat{f}^0|^2 \, dx \right)^{1/2} + \frac{\Delta\tau}{2} \langle z^1 + (\Lambda\mathbf{f}^k)^0, \hat{f}^0 \rangle \quad - \int_\Omega \nabla f^{0,k} \cdot \nabla \hat{f}^0 \, dx \Bigg], \quad (6.144)$$

$$\min_{\hat{f}^1 \in L^2(\Omega)} \Bigg[\frac{1}{2} \int_\Omega |\hat{f}^1|^2 \, dx + \beta_0 \frac{\Delta\tau}{2} \|\hat{f}^1\|_{L^2(\Omega)} - \frac{\Delta\tau}{2} \int_\Omega (z^0 - (\Lambda\mathbf{f}^k)^1)\hat{f}^1 \, dx - \int_\Omega f^{1,k} \hat{f}^1 \, dx \Bigg]. \quad (6.145)$$

Both problems (6.144), (6.145) have *closed form* solutions which can be obtained as in Section 1.8.8 for the solution of problem (1.115). The solution of problem (6.144) (respectively (6.145)) clearly provides the first (respectively the second) component of $\mathbf{f}^{k+1/2}$, i.e. the one in $H_0^1(\Omega)$ (respectively in $L^2(\Omega)$).

6.9. Experimental validation of the filtering procedure of Section 6.8.7 via the solution of the test problem of Section 6.8.5

We consider in this section the solution of the test problem of Section 6.8.5. The filtering technique discussed in Section 6.8.7 is applied with $\mathcal{T}_h$ a regular triangulation like the one shown on Figure 1 of Section 2.6; we recall that $\mathcal{T}_h$ is used to approximate $\mathbf{f}_h^{\Delta t}$, while ψ and φ are approximated on $\mathcal{T}_{h/2}$ as shown in Section 6.8.7. Instead of taking h to be equal to the length of the largest edges of $\mathcal{T}_h$, it is convenient here to take h as the length of the edges adjacent to the right angles of $\mathcal{T}_h$. The approximate problems (6.106) have been solved by the conjugate gradient algorithm (6.116)–(6.136) of Section 6.8.7.3. This algorithm has been *initialized* with $f_0^0 = f_0^1 = 0$ and we have used

$$\int_\Omega |\nabla g_k^0|^2 \,\mathrm{d}x + (g_k^1, g_k^1)_h \Big/ \int_\Omega |\nabla g_0^0|^2 \,\mathrm{d}x + (g_0^1, g_0^1)_h \leq 10^{-14} \tag{6.146}$$

as the *stopping criterion* (for calculations on a CRAY X-MP).

Let us also mention that the functions z^0, z^1, u of the test problem in Section 6.8.5, satisfy

$$\|z^0\|_{L^2(\Omega)} = 12.92\ldots, \quad \|z^1\|_{H^{-1}(\Omega)} = 11.77\ldots, \quad \|u\|_{L^2(\Sigma)} = 7.386\,68\ldots.$$

In the following we shall denote by $\|\cdot\|_{0,\Omega}, |\cdot|_{1,\Omega}, \|\cdot\|_{-1,\Omega}, \|\cdot\|_{0,\Sigma}$ the $L^2(\Omega)$, $H_0^1(\Omega)$, $H^{-1}(\Omega)$, $L^2(\Sigma)$ norms, respectively (here $|v|_{1,\Omega} = (\int_\Omega |\nabla v|^2 \,\mathrm{d}x)^{1/2}$ and $\|v\|_{-1,\Omega)} = |w|_{1,\Omega}$ where $w \in H_0^1(\Omega)$ is the solution of the Dirichlet problem $-\Delta w = v$ in Ω, $w = 0$ on Γ).

To approximate problem (6.50) by the discrete problem (6.106) we have been using $h = 1/4, 1/8, 1/16, 1/32, 1/64$ and $\Delta t = h/2\sqrt{2}$ (since the *wave equations* are solved on a space/time grid of step size $h/2$ for the space discretization and $h/2\sqrt{2}$ for the time discretization); we recall that $T = 15/4\sqrt{2}$.

Results of our numerical experiments have been summarized in Table 7. In this table f_*^0, f_*^1, u_* are defined as in Section 6.8.5, and the new quantities z_*^0, z_*^1 are the discrete analogues of $y(T)$ and $y_t(T)$, where y is the solution of $(6.33)_2$, associated via $(6.33)_1$, to the solution $\mathbf{f}$ of problem (6.50).

Remark 6.20 In Table 7 we have taken $h/2$ as discretization parameter to make easier comparisons with the results of Table 6 and Glowinski *et al.* (1990, Section 10).

Comparing the above results to those in Table 6, the following facts appear quite clearly.

1 The *filtering method* described in Section 6.8.7 has been a *very effective* cure to the *ill-posedness* of the approximate problem (6.88).
2 The number of *conjugate gradient* iterations necessary to achieve the convergence is (for h sufficiently small) essentially *independent* of h; in fact, if one realizes that for $h = 1/64$ the number of unknowns is $2 \times (63)^2 = 7938$, converging in 12 iterations is a fairly good performance.
3 The target functions z^0 and z^1 have been reached within a fairly high accuracy.

Table 7. *Table 2.1. Summary of numerical results.* [a] *indicates the number of conjugate gradient iterations.*

	$h/2$				
	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
a	7	10	12	12	12
CPU time(s)					
CRAY X-MP	0.1	0.6	2.8	14.8	83.9
$\frac{\Vert f^0-f^0_*\Vert_{0,\Omega}}{\Vert f^0\Vert_{0,\Omega}}$	9.6×10^{-2}	2.6×10^{-2}	2.2×10^{-2}	6.4×10^{-3}	1.5×10^{-3}
$\frac{\vert f^0-f^0_*\vert_{1,\Omega}}{\vert f^0\vert_{1,\Omega}}$	3.5×10^{-1}	1.8×10^{-1}	9×10^{-2}	4.4×10^{-2}	2.2×10^{-2}
$\frac{\Vert f^1-f^1_*\Vert_{0,\Omega}}{\Vert f^1\Vert_{0,\Omega}}$	1×10^{-1}	2.6×10^{-2}	1.5×10^{-2}	7×10^{-3}	3.2×10^{-3}
$\frac{\Vert z^0-z^0_*\Vert_{0,\Omega}}{\Vert z^0\Vert_{0,\Omega}}$	2.4×10^{-8}	3×10^{-8}	6×10^{-8}	8.3×10^{-8}	6.6×10^{-8}
$\frac{\Vert z^1-z^1_*\Vert_{-1,\Omega}}{\Vert z^1\Vert_{-1,\Omega}}$	6.9×10^{-7}	4.6×10^{-7}	9.4×10^{-6}	2×10^{-5}	8.5×10^{-5}
$\frac{\Vert u-u_*\Vert_{0,\Sigma}}{\Vert u\Vert_{0,\Sigma}}$	1.2×10^{-1}	4.3×10^{-2}	2×10^{-2}	7.6×10^{-3}	3.4×10^{-3}
$\Vert u_*\Vert_{0,\Sigma}$	7.271	7.386	7.453	7.405	7.381

The results of Table 6 compare favourably with those displayed in Tables 10.3 and 10.4 of Glowinski *et al.* (1990, pp. 58, 59) which were obtained using the Tychonoff regularization procedure briefly recalled in Section 6.8.6; in fact, fewer iterations are needed here, implying a smaller CPU time (actually the CPU time seems to be a *sublinear* function of h^{-3} which is – modulo a multiplicative constant – the number of points of the space/time discretization grid). Table 7 also shows that the approximation errors (roughly) satisfy

$$\|f^0 - f^0_*\|_{L^2(\Omega)} = 0(h^2), \quad \|f^0 - f^0_*\|_{H^1_0(\Omega)} = 0(h), \quad \|f^1 - f^1_*\|_{L^2(\Omega)} = 0(h), \tag{6.147}$$

$$\|u - u_*\|_{L^2(\Sigma)} = 0(h). \tag{6.148}$$

Estimates (6.147) are of *optimal order* with respect to h in the sense that they have the order that we can expect when one approximates the solution of a boundary value problem, for a second-order elliptic operator, by piecewise linear FE approximations; this result is not surprising since (from Section 6.8.2, relation (6.54)) the operator Λ associated with $\Omega = (0,1)\times(0,1)$ behaves for T *sufficiently large* like

$$2T\begin{pmatrix} -\Delta & 0 \\ 0 & I \end{pmatrix} \tag{6.149}$$

(we have here $x_0 = \{1/2, 1/2\}$ and $C = \frac{1}{2}$).

In order to visualize the influence of h we have plotted for $h = 1/4$, 1/8, 1/16, 1/32, 1/64 and $\Delta t = h/2\sqrt{2}$ the exact solutions f^0, f^1 and

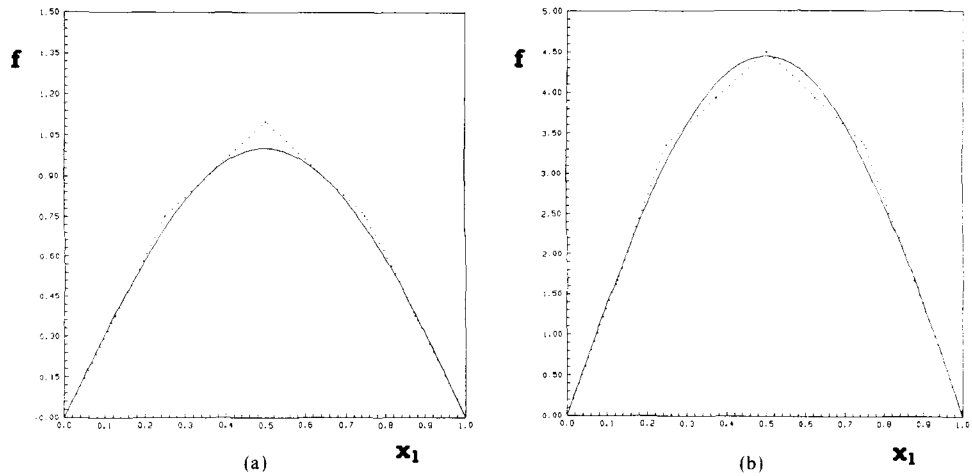


Fig. 54. ($h = 1/4$, $\Delta t = h/2\sqrt{2}$). (a) Variation of $f^0(x_1, 1/2)$ (——) and $f^0_*(x_1, 1/2)$ (······); (b) Variation of $-f^1(x_1, 1/2)$ (——) and $-f^1_*(x_1, 1/2)$ (······).

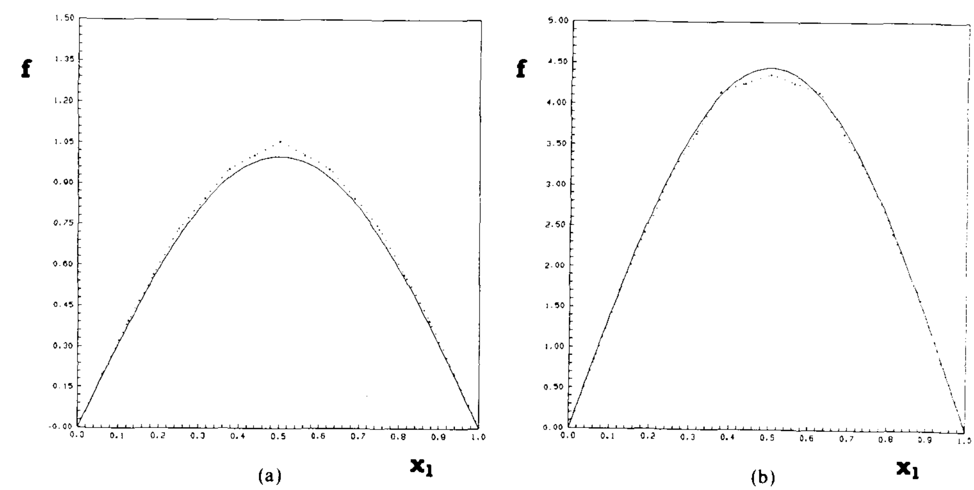


Fig. 55. ($h = 1/8$, $\Delta t = h/2\sqrt{2}$). (a) Variation of $f^0(x_1, 1/2)$ (——) and $f^0_*(x_1, 1/2)$ (······). (b) Variation of $-f^1(x_1, 1/2)$ (——) and $-f^1_*(x_1, 1/2)$ (······).

the corresponding computed solutions f^0_*, f^1_*. To be more precise, we have shown the plots of the functions $x_1 \to f^0(x_1, 1/2)$, $x_1 \to -f^1(x_1, 1/2)$ (full curves) and of the corresponding computed functions (dotted curves). These results have been reported in Figures 54 to 58 and the captions there are self-explanatory.

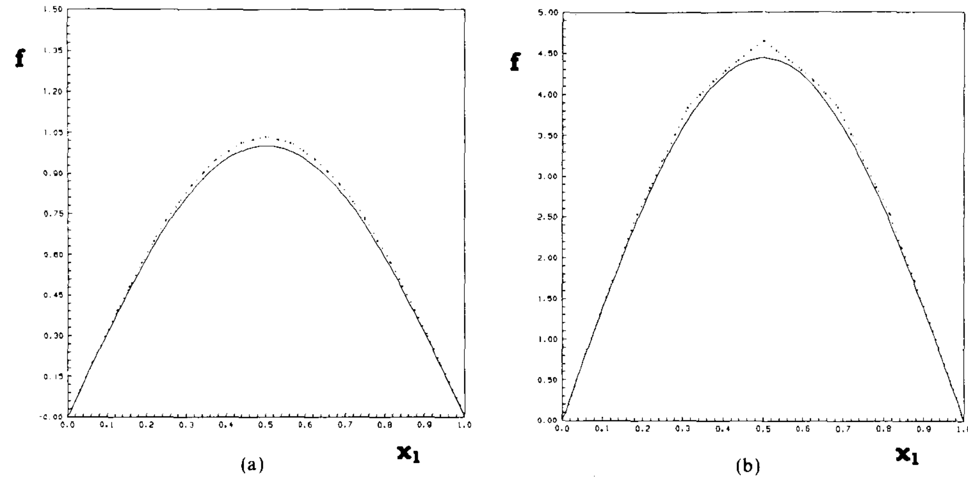


Fig. 56. ($h = 1/16$, $\Delta t = h/2\sqrt{2}$). (a) Variation of $f^0(x_1, 1/2)$ (——) and $f_*^0(x_1, 1/2)$ $(\cdots\cdots)$. (b) Variation of $-f^1(x_1, 1/2)$ (——) and $-f_*^1(x_1, 1/2)$ $(\cdots\cdots)$.

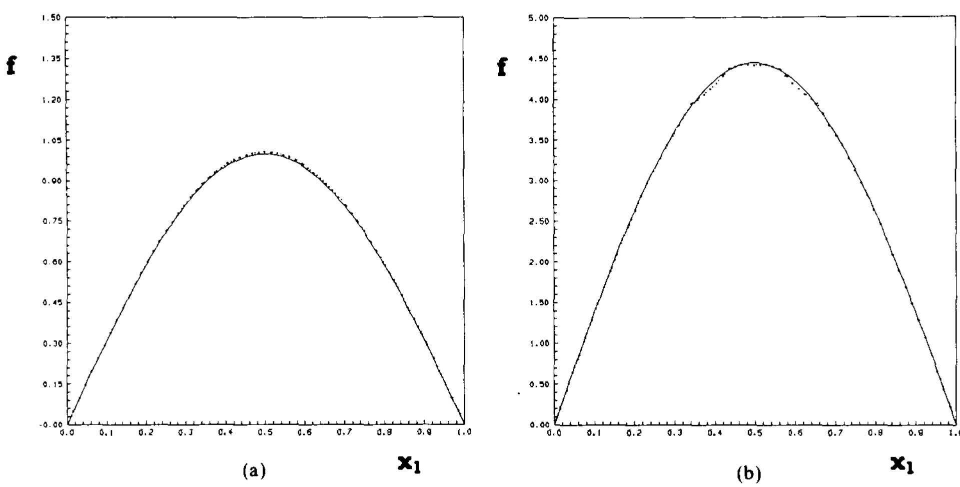


Fig. 57. ($h = 1/32$, $\Delta t = h/2\sqrt{2}$). (a) Variation of $f^0(x_1, 1/2)$ (——) and $f_*^0(x_1, 1/2)$ $(\cdots\cdots)$. (b) Variation of $-f^1(x_1, 1/2)$ (——) and $-f_*^1(x_1, 1/2)$ $(\cdots\cdots)$.

The above numerical experiments have been done with $T = 15/4\sqrt{2}$; in order to study the influence of T we have kept z^0 and z^1 as in the above experiments and taken $T = 28.2843$. For $h = 1/64$ and $\Delta t = h/2\sqrt{2}$ we need just 10 iterations of algorithm (6.116)–(6.136) to achieve convergence, the corresponding CRAY X-MP CPU time being then 800s (!) (the number

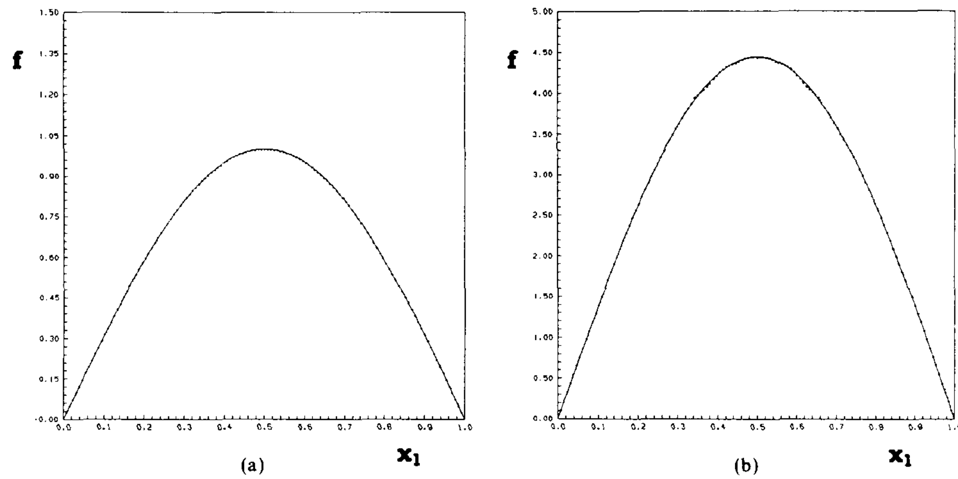


Fig. 58. ($h = 1/64$, $\Delta t = h/2\sqrt{2}$). (a) Variation of $f^0(x_1, 1/2)$ (——) and $f^0_*(x_1, 1/2)$ (······). (b) Variation of $-f^1(x_1, 1/2)$ (——) and $-f^1_*(x_1, 1/2)$ (······).

of grid points for the space/time discretization is now $\approx 86 \times 10^6$). We have $\|u_*\|_{L^2(\Sigma)} = 2.32$, $\|z^0 - z^0_*\|_{L^2(\Omega)} = 5.8 \times 10^{-6}$, $\|z^1 - z^1_*\|_{-1,\Omega} = 1.6 \times 10^{-5}$. The most interesting results are the ones reported on Figures 59(a) and (b). There, we have compared Tf^0_* and Tf^1_* (for $T = 28.2843$) with the corresponding theoretical limits χ^0 and χ^1 which, according to Section 6.8.2, relations (6.55)–(6.57), are given by

$$\Delta\chi^0 = z^1/2 \text{ in } \Omega, \quad \chi^0 = 0 \text{ on } \Gamma, \tag{6.150}$$

$$\chi^1 = z^0/2. \tag{6.151}$$

The *full* curves represent the variations of $x_1 \to \chi^0(x_1, 1/2)$ and of $x_1 \to -\chi^1(x_1, 1/2)$, while the *dotted* curves represent the variations of $x_1 \to Tf^0_*(x_1, 1/2)$ and $x_1 \to -Tf^1_*(x_1, 1/2)$.

In our opinion the above figures provide an excellent *numerical verification* of the convergence result (6.55) of Section 6.8.2 (we observe at $x_1 = 0$ and $x_1 = 1$ a (numerical) *Gibbs phenomenon* associated with the L^2 convergence of Tf^1_* to χ^1). Conversely, these results provide a *validation* of the numerical methodology discussed here; they show that this methodology is particularly robust, accurate, nondissipative and perfectly able to handle very long time intervals $[0, T]$. In fact, numerical experiments have shown that the above-mentioned qualities of the numerical methods discussed here persist for target functions z^0 and z^1 much rougher than those considered in this section.

Additional results can be found in Glowinski *et al.* (1990, Section 4).

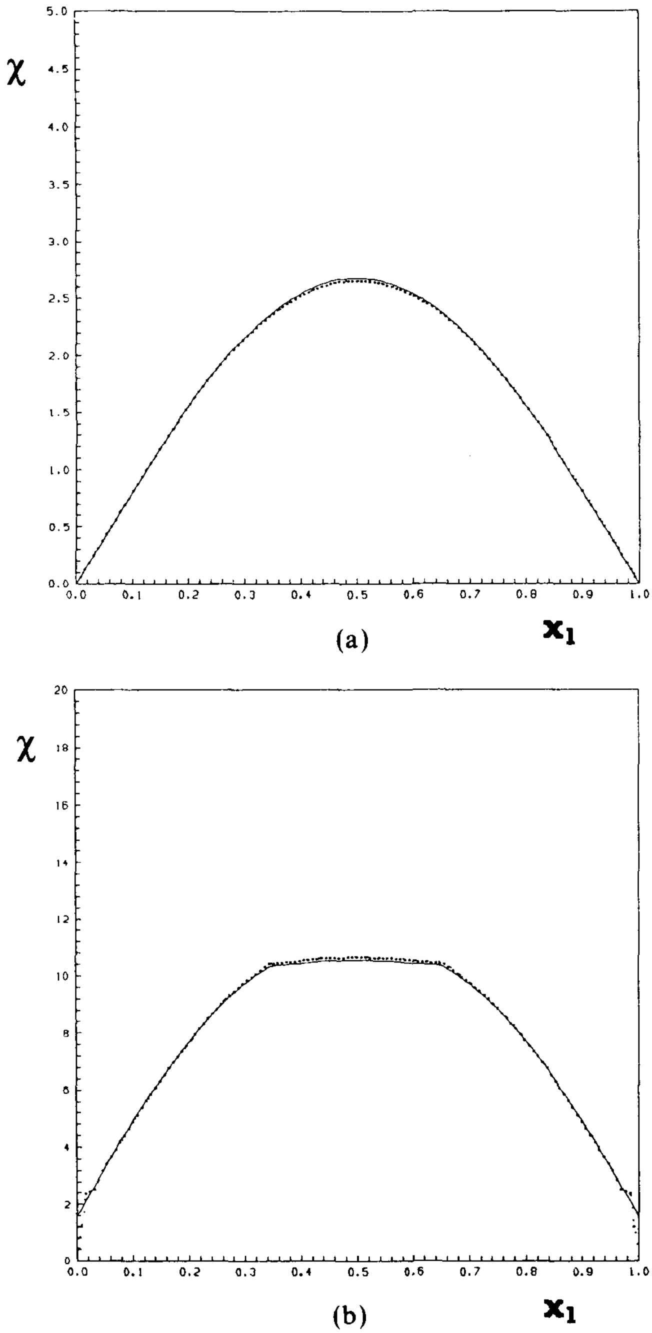


Fig. 59. ($h = 1/64$, $\Delta t = h/2\sqrt{2}$, $T = 28.2843$). (a) Variation of $\chi^0(x_1, 1/2)$ (——) and $Tf^0_*(x_1, 1/2)$ $(\cdots\cdots)$. (b) Variation of $-\chi^1(x_1, 1/2)$ (——) and $-Tf^1_*(x_1, 1/2)$ $(\cdots\cdots)$.

6.10. Other boundary controls

6.10.1. Approximate Neumann boundary controllability

We consider now problems entirely similar to the previous ones but where we 'exchange' the *Dirichlet conditions* for *Neumann conditions.*

We therefore define the state function $y = y(v)$ by

$$y_{tt} + Ay = 0 \text{ in } Q = \Omega \times (0, T), \tag{6.152$_1$}$$

$$y(0) = 0, \quad y_t(0) = 0, \tag{6.152$_2$}$$

$$\frac{\partial y}{\partial n_A} = v \text{ on } \Sigma_0 = \Gamma_0 \times (0, T), \quad \frac{\partial y}{\partial n_A} = 0 \text{ on } \Sigma \backslash \Sigma_0. \tag{6.152$_3$}$$

To fix ideas, we still assume that

$$v \in L^2(\Sigma_0). \tag{6.153}$$

Using again *transposition*, we can show that problem (6.152) has a *unique* (weak) solution if (6.153) holds. In fact, the solution here is (slightly) smoother than the one in Section 6.1. We can, in any case, define an operator L from $L^2(\Sigma_0)$ into $H^{-1}(\Omega) \times L^2(\Omega)$ by

$$Lv = \{-y_t(T; v), y(T; v)\}. \tag{6.154}$$

If v is smooth, the solution $y(v)$ will also be smooth, assuming of course that the coefficients of A are also smooth.

Let us study *approximate controllability* first.

We suppose that v is *smooth*; indeed, to fix ideas we assume that

$$v, \frac{\partial v}{\partial t} \in L^2(\Sigma_0), \quad v|_t = 0. \tag{6.155}$$

Then, $y(v)$ can be defined by a *variational formulation*, showing that

$$\begin{cases} y & \text{is continuous from } [0, T] \text{ into } H^1(\Omega), \\ y_t & \text{is continuous from } [0, T] \text{ into } L^2(\Omega). \end{cases} \tag{6.156}$$

Then in particular

$$Lv \in L^2(\Omega) \times L^2(\Omega). \tag{6.157}$$

Let us consider now $\mathbf{f}$ belonging to the orthogonal of the range of L, i.e.

$$\begin{cases} \mathbf{f} = \{f^0, f^1\} \in L^2(\Omega) \times L^2(\Omega), \\ -\int_\Omega f^0 y_t(T)\, \mathrm{d}x + \int_\Omega f^1 y(T)\, \mathrm{d}x = 0, \ \forall v \text{ satisfying (6.155).} \end{cases} \tag{6.158}$$

We introduce ψ defined by

$$\psi_{tt} + A\psi = 0 \text{ in } Q, \quad \psi(T) = f^0, \ \psi_t(T) = f^1, \ \frac{\partial \psi}{\partial n_A} = 0 \text{ on } \Sigma. \tag{6.159}$$

Multiplying the wave equation in (6.159) by $y = y(v)$ and integrating by parts, we obtain

$$\langle Lv, \mathbf{f} \rangle = -\int_{\Sigma_0} \psi v \, d\Sigma. \tag{6.160}$$

Therefore (6.158) is equivalent to

$$\psi = 0 \text{ on } \Sigma_0. \tag{6.161}$$

If we assume (as in Section 6.3, relation (6.22)) that

Σ_0 allows the application of the Holmgren's uniqueness theorem (6.162)

then (6.159), (6.161) imply that $\psi = 0$ in Q, so that $f = 0$; we have proved thus that

assuming (6.162) the range of L is dense in $L^2(\Omega) \times L^2(\Omega)$, (6.163)

which implies, in turn, *approximate controllability.*

Remark 6.20 Suppose that Γ is a C^∞ manifold. Then we can take

$v \in \mathcal{D}(\Sigma_0)$ (the space of the C^∞ functions with compact support in Σ_0), (6.164)

and the range of L, for v describing $\mathcal{D}(\Sigma_0)$, is still dense in $L^2(\Omega) \times L^2(\Omega)$.

We can now state the following control problem

$$\inf_v \frac{1}{2} \int_{\Sigma_0} v^2 \, d\Sigma; y(T) \in z^0 + \beta_0 B, \quad y_t(T) \in z^1 + \beta_1 B, \tag{6.165}$$

where, in (6.165), $\{y, v\}$ satisfies (6.152), (6.153), $\{z^0, z^1\} \in L^2(\Omega) \times L^2(\Omega)$, and where B denotes the closed unit ball of $L^2(\Omega)$.

Remark 6.21 We do *not* introduce $H^{-1}(\Omega)$ here for two reasons:

1. in the present context the $H^{-1}(\Omega)$ space (which is not the dual of $H^1(\Omega)$) is not natural;
2. the choice of the same norm, in (6.165), for both $y(T)$ and $y_t(T)$ shows the flexibility of the methodology.

Remark 6.22 Problem (6.165) has a unique solution. *Uniqueness* follows from the *strict convexity.* As far as *existence* is concerned let $\{u_n\}_{n\geq 0}$ be a *minimizing sequence.* Then $\{u_n\}_{n\geq 0}$ is *bounded* in $L^2(\Sigma_0)$. Let us set $y_n = y(u_n)$. By definition, $\{\{y_n(T), \partial y_n/\partial t(T)\}\}_{n\geq 0}$ remains in a bounded set of $L^2(\Omega) \times L^2(\Omega)$. We can therefore extract from $\{u_n\}_{n\geq 0}$ a subsequence, still denoted by $\{u_n\}_{n\geq 0}$, such that

$$u_n \to u \;\text{ weakly in } L^2(\Sigma_0), \tag{6.166}$$

$$\left\{ y_n(T), \frac{\partial y_n}{\partial t}(T) \right\} \to \{\xi_0, \xi_1\} \;\text{ weakly in } L^2(\Omega) \times L^2(\Omega). \tag{6.167}$$

However,

$$\{y(T;u_n), y_t(T;u_n)\} \to \{y(T;u), y_t(T;u)\}$$

weakly in $L^2(\Omega) \times H^{-1}(\Omega)$, so that $\xi_0 = y(T;u), \xi_1 = y_t(T;u)$, which proves the existence of a solution namely u to problem (6.165).

The uniqueness of the solution implies that the *whole* minimizing sequence converges to u.

Remark 6.23 The use of β_0 and $\beta_1 > 0$ allows the introduction of new (and complicated) function spaces to be avoided. Unfortunately, these spaces cannot be avoided if we let β_0 and $\beta_1 \to 0$, as we shall see below.

6.10.2. Duality results: exact Neumann boundary controllability

Now, we use *duality*, as in previous sections. We then introduce functionals F_1 and F_2 by

$$F_1(v) = \frac{1}{2}\int_{\Sigma_0} v^2 \, d\Sigma, \tag{6.168}$$

$$F_2(\hat{\mathbf{f}}) = \begin{cases} 0 \text{ if } \hat{f}^0 \in -z^1 + \beta_1 B \quad \hat{f}^1 \in z^0 + \beta_0 B, \\ +\infty \ \text{ otherwise on } L^2(\Omega) \times L^2(\Omega). \end{cases} \tag{6.169}$$

It follows then from (6.154) that problem (6.165) is equivalent to

$$\inf_{v \in L^2(\Sigma_0)} [F_1(v) + F_2(Lv)]. \tag{6.170}$$

Using *convex duality* arguments, we obtain

$$\inf_{v \in L^2(\Sigma_0)} [F_1(v) + F_2(Lv)] = - \inf_{\hat{\mathbf{f}} \in L^2(\Omega)\times L^2(\Omega)} [F_1^*(L^*\hat{\mathbf{f}}) + F_2^*(-\hat{\mathbf{f}})], \tag{6.171}$$

where we use L^* with L thought of as an unbounded operator.

By virtue of (6.171), we have

$$L^*\hat{\mathbf{f}} = -\hat{\psi}|_{\Sigma_0}, \tag{6.172}$$

where $\hat{\psi}$ is the solution of (6.159) when $\hat{\mathbf{f}}$ replaces $\mathbf{f}$.

We obtain then as *dual problem* of the control problem (6.165)

$$\inf_{\hat{\mathbf{f}} \in L^2(\Omega)\times L^2(\Omega)} \left[\frac{1}{2}\int_{\Sigma_0} \hat{\psi}^2 \, d\Sigma + \int_\Omega (z^1 \hat{f}_0 - z^0 \hat{f}^1) \, dx + \beta_1 \|\hat{f}^0\|_{L^2(\Omega)} + \beta_0 \|\hat{f}^1\|_{L^2(\Omega)}\right]. \tag{6.173}$$

Remark 6.23 We shall give an alternative formulation of the dual problem (6.173). This new formulation is particularly useful when $\{\beta_0, \beta_1\} \to \mathbf{0}$. Using the HUM approach, we introduce the operator Λ defined as follows.

The functions $\hat{f}^0$ and $\hat{f}^1$ being given in, say, $L^2(\Omega)$, we define $\hat{\psi}$ and $\hat{y}$ by

$$\hat{\psi}_{tt} + A\hat{\psi} = 0 \text{ in } Q, \quad \hat{\psi}(T) = \hat{f}^0,\ \hat{\psi}_t(T) = \hat{f}^1,\ \frac{\partial \hat{\psi}}{\partial n_A} = 0 \text{ on } \Sigma, \tag{6.174}$$

$$\begin{cases} \hat{y}_{tt} + A\hat{y} = 0 \text{ in } Q, \quad \hat{y}(0) = \hat{y}_t(0) = 0, \quad \dfrac{\partial \hat{y}}{\partial n_A} = -\hat{\psi} \text{ on } \Sigma_0, \\ \dfrac{\partial \hat{y}}{\partial n_A} = 0 \text{ on } \Sigma \backslash \Sigma_0. \end{cases} \tag{6.175}$$

We set then (with $\hat{\mathbf{f}} = \{\hat{f}^0, \hat{f}^1\}$):

$$\Lambda\hat{\mathbf{f}} = \{-\hat{y}_t(T), \hat{y}(T)\}. \tag{6.176}$$

Taking $\hat{\mathbf{f}} = \mathbf{f}_1$ and $\mathbf{f}_2$, and denoting by ψ_1, ψ_2 the corresponding solutions of (6.174) we obtain from (6.174), (6.175) that

$$\int_\Omega (\Lambda \mathbf{f}_1) \cdot \mathbf{f}_2 \, \mathrm{d}x = \int_{\Sigma_0} \psi_1 \psi_2 \, \mathrm{d}\Sigma. \tag{6.177}$$

It follows from (6.177) that

$$\Lambda \text{ is symmetric and positive semi-definite over } L^2(\Omega) \times L^2(\Omega). \tag{6.178}$$

It follows from (6.177) that problem (6.173) is *equivalent* to

$$\inf_{\hat{\mathbf{f}} \in L^2(\Omega) \times L^2(\Omega)} \left[\frac{1}{2} \int_\Omega (\Lambda \hat{\mathbf{f}}) \cdot \hat{\mathbf{f}} \, \mathrm{d}x + \int_\Omega (z^1 \hat{f}^0 - z^0 \hat{f}^1) \, \mathrm{d}x + \beta_1 \|\hat{f}^0\|_{L^2(\Omega)} \right.$$
$$\left. + \beta_0 \|\hat{f}^1\|_{L^2(\Omega)} \right]. \tag{6.179}$$

In order to discuss the case $\beta_0 = \beta_1 = 0$ in (6.179), we introduce over $L^2(\Omega) \times L^2(\Omega)$ the norm [...] defined by

$$[\hat{\mathbf{f}}] = \left(\int_\Omega (\Lambda \hat{\mathbf{f}}) \cdot \hat{\mathbf{f}} \, \mathrm{d}x \right)^{1/2}, \ \forall \hat{\mathbf{f}} \in L^2(\Omega) \times L^2(\Omega). \tag{6.180}$$

We define next the space E by

$$E = \text{ completion of } L^2(\Omega) \times L^2(\Omega) \text{ for the norm } [\ldots]. \tag{6.181}$$

Taking now the limit in (6.179) as $\{\beta_0, \beta_1\} \to \mathbf{0}$ we obtain – *formally* – the dual problem associated with *exact controllability*, namely

$$\inf_{\hat{\mathbf{f}} \in E} \left[\frac{1}{2} [\hat{\mathbf{f}}]^2 - \langle \sigma \mathbf{z}, \hat{\mathbf{f}} \rangle \right] \tag{6.182}$$

where, in (6.182), $\langle \ldots, \ldots \rangle$ denotes the duality pairing between E' and E, $\mathbf{z} = \{z^0, z^1\}$ and $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Problem (6.182) *has a solution (necessarily unique) if and only if*

$$\{-z^1, z^0\} \in E'; \tag{6.183}$$

equivalently, exact controllability is true if and only if condition (6.183) *is satisfied.*

Remark 6.24 Contrary to the situation in Section 6.6, the space E, as defined by (6.181), has no simple interpretation. For further information concerning space E, we refer to Lions (1988b) and the references therein.

Remark 6.25 It is by now clear that the method followed here is general. It can, in particular, be applied to *other boundary conditions.*

6.10.3. A second approximate Neumann boundary controllability problem

Inspired by Sections 1 and 2, we consider, for $k > 0$, the following control problem

$$\min_{v \in L^2(\Sigma_0)} \left[\frac{1}{2}\int_{\Sigma_0} v^2 \, d\Sigma + \frac{k}{2}(\|y(T) - z^0\|^2_{L^2(\Omega)} + \|y_t(T) - z^1\|^2_{L^2(\Omega)})\right], \tag{6.184}$$

where, in (6.184), y is – still – defined from v via the wave equation (6.152); problem (6.184) is obtained by *penalization* of the conditions $y(T) = z^0$, $y_t(T) = z^1$.

Using the results of Section 6.10.1 it is quite easy to show that *problem* (6.184) *has a (necessarily unique) solution* (even if (6.162) does not hold). If we denote by u the solution of problem (6.184), it is characterized by the existence of an *adjoint state* function p such that

$$\left\{\begin{array}{l} y_{tt} + Ay = 0 \text{ in } Q, \quad y(0) = y_t(0) = 0, \\ \dfrac{\partial y}{\partial n_A} = u \text{ on } \Sigma_0, \quad \dfrac{\partial y}{\partial n_A} = 0 \text{ on } \Sigma\backslash\Sigma_0, \end{array}\right. \tag{6.185}$$

$$p_{tt} + Ap = 0 \text{ in } Q, \quad \frac{\partial p}{\partial n_A} = 0 \text{ on } \Sigma, \tag*{(6.186)$_1$}$$

$$p(T) = k(y_t(T) - z^1), \quad p_t(T) = -k(y(T) - z^0), \tag*{(6.186)$_2$}$$

$$u = -p \text{ on } \Sigma_0. \tag{6.187}$$

Let us define $\mathbf{f} = \{f^0, f^1\} \in L^2(\Omega) \times L^2(\Omega)$ by

$$f^0 = p(T), \quad f^1 = p_t(T); \tag{6.188}$$

it follows then from $(6.186)_2$, and from the definition of Λ (see Section 6.10.2) that

$$k^{-1}\mathbf{f} + \Lambda\mathbf{f} = \{-z^1, z^0\}. \tag{6.189}$$

Problem (6.189) is the *dual* problem of (6.184).

From the properties of Λ (*symmetry* and *positivity*) and from the $(L^2(\Omega))^2$ ellipticity of the bilinear form associated with operator $k^{-1}I + \Lambda$, problem (6.189) can be solved by a *conjugate gradient algorithm* operating in $L^2(\Omega) \times L^2(\Omega)$; such an algorithm will be described in Section 6.10.4.

6.10.4. Conjugate gradient solution of the dual problem (6.189)

We can solve problem (6.189) by the following variant of algorithm (6.58)–(6.73) (see Section 6.8.3):

$$\mathbf{f}_0 = \{f_0^0, f_0^1\} \text{ given in } L^2(\Omega) \times L^2(\Omega); \tag{6.190}$$

solve then

$$\begin{cases} \dfrac{\partial^2 \psi_0}{\partial t^2} + A\psi_0 = 0 \text{ in } Q, \quad \psi_0(T) = f_0^0, \\ \dfrac{\partial \psi_0}{\partial t}(T) = f_0^1, \quad \dfrac{\partial \psi_0}{\partial n_A} = 0 \text{ on } \Sigma, \end{cases} \tag{6.191}$$

$$\begin{cases} \dfrac{\partial^2 \varphi_0}{\partial t^2} + A\varphi_0 = 0 \text{ in } Q, \quad \varphi_0(0) = \dfrac{\partial \varphi_0}{\partial t}(0) = 0, \\ \dfrac{\partial \varphi_0}{\partial n_A} = -\psi_0 \text{ on } \Sigma_0, \quad \dfrac{\partial \varphi_0}{\partial n} = 0 \text{ on } \Sigma \backslash \Sigma_0. \end{cases} \tag{6.192}$$

Define $\mathbf{g}_0 = \{g_0^0, g_0^1\} \in L^2(\Omega) \times L^2(\Omega)$ *by*

$$\int_\Omega g_0^0 v \,\mathrm{d}x = k^{-1} \int_\Omega f_0^0 v \,\mathrm{d}x + \int_\Omega \left(z^1 - \frac{\partial \varphi_0}{\partial t}(T)\right) v \,\mathrm{d}x, \ \forall v \in L^2(\Omega), \tag*{(6.193)$_1$}$$

$$\int_\Omega g_0^1 v \,\mathrm{d}x = k^{-1} \int_\Omega f_0^1 v \,\mathrm{d}x + \int_\Omega (\varphi_0(T) - z^0) v \,\mathrm{d}x, \ \forall v \in L^2(\Omega), \tag*{(6.193)$_2$}$$

and define $\mathbf{w}_0 = \{w_0^0, w_1^0\}$ *by*

$$\mathbf{w}_0 = \mathbf{g}_0. \tag{6.194}$$

Assuming that $\mathbf{f}_n, \mathbf{g}_n, \mathbf{w}_n$ *are known, we obtain* $\mathbf{f}_{n+1}, \mathbf{g}_{n+1}, \mathbf{w}_{n+1}$ *as follows.*

Solve

$$\begin{cases} \dfrac{\partial^2 \bar{\psi}_n}{\partial t^2} + A\bar{\psi}_n = 0 \text{ in } Q, \quad \bar{\psi}_n(T) = w_n^0, \\ \dfrac{\partial \bar{\psi}_n}{\partial t}(T) = w_n^1, \quad \dfrac{\partial \bar{\psi}_n}{\partial n_A} = 0 \text{ on } \Sigma, \end{cases} \tag{6.195}$$

$$\begin{cases} \dfrac{\partial^2 \bar{\varphi}_n}{\partial t^2} + A\bar{\varphi}_n = 0 \text{ in } Q, \quad \bar{\varphi}_n(0) = \dfrac{\partial \bar{\varphi}_n}{\partial t}(0) = 0, \\ \dfrac{\partial \bar{\varphi}_n}{\partial n_A} = -\bar{\psi}_n \text{ on } \Sigma_0, \quad \dfrac{\partial \bar{\varphi}_n}{\partial n_A} = 0 \text{ on } \Sigma \backslash \Sigma_0. \end{cases} \tag{6.196}$$

Define $\bar{\mathbf{g}}_n = \{\bar{g}_n^0, \bar{g}_n^1\} \in L^2(\Omega) \times L^2(\Omega)$ *by*

$$\int_\Omega \bar{g}_n^0 v \,\mathrm{d}x = k^{-1} \int_\Omega w_n^0 v \,\mathrm{d}x - \int_\Omega \frac{\partial \bar{\varphi}_n}{\partial t}(T) v \,\mathrm{d}x, \ \forall v \in L^2(\Omega), \tag{6.197$_1$}$$

$$\int_\Omega \bar{g}_n^1 v \,\mathrm{d}x = k^{-1} \int_\Omega w_n^1 v \,\mathrm{d}x + \int_\Omega \bar{\varphi}_n(T) v \,\mathrm{d}x, \ \forall v \in L^2(\Omega). \tag{6.197$_2$}$$

Compute

$$\rho_n = \int_\Omega (|g_n^0|^2 + |g_n^1|^2) \,\mathrm{d}x \Big/ \int_\Omega (\bar{g}_n^0 w_n^0 + \bar{g}_n^1 w_n^1) \,\mathrm{d}x, \tag{6.198}$$

and then

$$\mathbf{f}_{n+1} = \mathbf{f}_n - \rho_n \mathbf{w}_n, \tag{6.199}$$

$$\mathbf{g}_{n+1} = \mathbf{g}_n - \rho_n \bar{\mathbf{g}}_n. \tag{6.200}$$

If $\|\mathbf{g}_{n+1}\|_{L^2(\Omega)\times L^2(\Omega)} / \|\mathbf{g}_0\|_{L^2(\Omega)\times L^2(\Omega)} \leq \epsilon$ *take* $\mathbf{f} = \mathbf{f}_{n+1}$; *if not compute*

$$\gamma_n = \frac{\|\mathbf{g}_{n+1}\|^2_{L^2(\Omega)\times L^2(\Omega)}}{\|\mathbf{g}_n\|^2_{L^2(\Omega)\times L^2(\Omega)}} \tag{6.201}$$

and update $\mathbf{w}_n$ *by*

$$\mathbf{w}_{n+1} = \mathbf{g}_{n+1} + \gamma_n \mathbf{w}_n. \tag{6.202}$$

Do $n = n + 1$ *and go to* (6.195).

Remark 6.26 The FE implementation of the above algorithm is just a variation of the one of algorithm (6.58)–(6.73) (it is in fact simpler). In fact, here too we can take advantage of the *reversibility* of the wave equations to reduce the storage requirements of the discrete analogues of algorithm (6.190)–(6.202).

Remark 6.27 In Glowinski and Li (1990), we can find a discussion of numerical methods for solving exact Neumann boundary controllability problems; the solution method is based on a combination of *finite element* approximations and of a conjugate gradient algorithm closely related to algorithm (6.190)–(6.202). We also discuss, in the above reference, the asymptotic behaviour of the solution $\mathbf{f}$ of the dual problem when $T \to +\infty$; there too the analytical results confirmed the numerical ones, validating therefore the computational methodology.

6.10.5. Application to the solution of the dual problem (6.179)

Assuming that β_0 and β_1 are *positive* the dual problem (6.179) can also be written as

$$\Lambda \mathbf{f} + \partial j(\mathbf{f}) = \begin{pmatrix} -z^1 \\ z^0 \end{pmatrix}, \tag{6.203}$$

where the functional $j : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ is defined by

$$j(\hat{\mathbf{f}}) = \beta_1 \|\hat{f}^0\|_{L^2(\Omega)} + \beta_0 \|\hat{f}^1\|_{L^2(\Omega)}, \ \forall \hat{\mathbf{f}} = \{f^0, f^1\} \in L^2(\Omega) \times L^2(\Omega). \quad (6.204)$$

As in Section 6.8.8, we associate with (6.203) the following *initial value problem*

$$\begin{cases} \dfrac{\partial \mathbf{f}}{\partial \tau} + \Lambda \mathbf{f} + \partial j(\mathbf{f}) = \begin{pmatrix} -z^1 \\ z^0 \end{pmatrix}, \\ \mathbf{f}(0) = \mathbf{f}_0 \end{cases} \quad (6.205)$$

to be discretized, for example, by the following *Peaceman–Rachford scheme*:

$$\mathbf{f}^0 = \mathbf{f}_0; \quad (6.206)$$

then for $k \geq 0$, assuming that $\mathbf{f}^k$ is known, we compute $\mathbf{f}^{k+1/2}$ and $\mathbf{f}^{k+1}$ via

$$\frac{\mathbf{f}^{k+1/2} - \mathbf{f}^k}{\Delta\tau/2} + \Lambda f^k + \partial j(\mathbf{f}^{k+1/2}) = \begin{pmatrix} -z^1 \\ z^0 \end{pmatrix}, \quad (6.207)$$

$$\frac{\mathbf{f}^{k+1} - \mathbf{f}^{k+1/2}}{\Delta\tau/2} + \Lambda f^{k+1} + \partial j(\mathbf{f}^{k+1/2}) = \begin{pmatrix} -z^1 \\ z^0 \end{pmatrix}. \quad (6.208)$$

Problem (6.207) is fairly easy to solve (see Section 6.8.8) since the operator $\partial j(\ldots)$ is *diagonal*. On the other hand, once $\mathbf{f}^{k+1/2}$ is known, problem (6.208) is just a particular case of problem (6.189) (with $k = \Delta\tau/2$); it can be solved therefore by the conjugate gradient algorithm (6.190)–(6.202).

6.11. Distributed controls for wave equations

Let us consider $\mathcal{O} \subset \Omega$ and let the *state equation* be

$$y_{tt} + Ay = v\chi_{\mathcal{O}} \text{ in } Q, \quad y(0) = y_t(0) = 0, \quad y = 0 \text{ on } \Sigma. \quad (6.209)$$

We choose

$$v \in L^2(\mathcal{O} \times (0,T)). \quad (6.210)$$

The solution of problem (11.1) is *unique*. It satisfies

$$\{y, y_t\} \text{ is continuous from } [0,T] \text{ into } H_0^1(\Omega) \times L^2(\Omega). \quad (6.211)$$

Let us see when

$$\{y(T), y_t(T)\} \text{ spans a dense subset of } H_0^1(\Omega) \times L^2(\Omega). \quad (6.212)$$

We consider $\mathbf{f} = \{f^0, f^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$ such that

$$-\int_\Omega y_t(T) f^0 \,\mathrm{d}x + \langle f^1, y(T)\rangle = 0, \ \forall v \in L^2(\mathcal{O} \times (0,T)), \quad (6.213)$$

where, in (6.213), $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

We introduce ψ solution of

$$\psi_{tt} + A\psi = 0 \text{ in } Q, \quad \psi(T) = f^0, \ \psi_t(T) = f^1, \ \psi = 0 \text{ on } \Sigma. \tag{6.214}$$

Then

$$-\int_\Omega y_t(T) f^0 \, dx + \langle f^1, y(T) \rangle = \int_{\mathcal{O}\times(0,T)} \psi v \, dx \, dt. \tag{6.215}$$

Therefore (6.213) is equivalent to

$$\psi = 0 \text{ on } \mathcal{O} \times (0,T). \tag{6.216}$$

We shall assume that we can apply Holmgren's uniqueness theorem to $\mathcal{O} \times (0,T)$; then $\psi \equiv 0$ and $f = 0$, so that (6.212) holds true.

We can then consider

$$\inf_v \frac{1}{2} \int\int_{\mathcal{O}\times(0,T)} v^2 \, dx \, dt; y(T;v) \in z^0 + \beta_0 B_1, y_t(T;v) \in z^1 + \beta_1 B \tag{6.217}$$

where, in (6.217), $y(v)$ is obtained from v via (6.209), $\{z^0, z^1\}$ is given in $H_0^1(\Omega) \times L^2(\Omega)$, B_1 (respectively B) is the closed unit ball of $H_0^1(\Omega)$ (respectively $L^2(\Omega)$).

Similar considerations to everything which has been said in the previous sections can be adapted to the present situation, from either the *purely mathematical point of view* (see Lions (1988b)) or the *numerical point of view.*

Remark 6.28 One can also consider *pointwise control*, as in

$$\begin{cases} y_{tt} + Ay = v(t)\delta(x-b) \text{ in } Q, \\ y(0) = y_t(0) = 0, \quad y = 0 \text{ on } \Sigma \text{ (to fix ideas)}. \end{cases} \tag{6.218}$$

Control problems for systems modelled by (6.218) have been discussed in Lions (1988b, Volume 1, Chapter 7). Interesting phenomena appear concerning the role of $b \in \Omega$. Methods from harmonic analysis have been used in this respect by Meyer (1989) and further developed by Haraux and Jaffard (1991), I. Joó (1991).

6.12. Dynamic Programming

We are going to apply *Dynamic Programming* to the situations described in Section 6.11. The approach is *formal*, somewhat similar to the one in Section 5.

Remark 6.29. We could have applied dynamic programming to the situations described in Sections 6.1 or 6.10, but the situation is simpler for the control problems described in Section 6.11.

We consider for s given in $(0,T)$

$$\begin{cases} y_{tt} + Ay = v\chi_{\mathcal{O}} \text{ in } \Omega \times (s,T), \\ y(s) = h^0, \quad y_t(s) = h^1, \quad y = 0 \text{ on } \Sigma_s = \Gamma \times (s,T), \end{cases} \tag{6.219}$$

with $\{h^0, h^1\} \in H_0^1(\Omega) \times L^2(\Omega)$.

We introduce

$$\phi(h^0, h^1, s) = \inf_v \int\int_{\mathcal{O}\times(s,T)} v^2 \, dx \, dt \tag{6.220}$$

where, in (6.220), v is such that $\{y(v), v\}$ satisfies (6.219) and

$$y(T; v) \in z^0 + \beta_0 B_1, \quad y_t(T; v) \in z^1 + \beta_1 B. \tag{6.221}$$

The quantity $\phi(h^0, h^1, s)$ is finite for every

$$z^0 \in H_0^1(\Omega), \; z^1 \in L^2(\Omega), \quad \beta_0 > 0, \; \beta_1 > 0$$

if and only if the Holmgren's uniqueness theorem applies for $\mathcal{O} \times (s, T)$ in $\Omega \times (s, T)$. *This is true for $s < s_0, s_0$ a suitable number in* $(0, T)$. In that case, the infimum in (6.221) is *finite* for $s < s_0$, implying that the function ϕ is *defined* over $H_0^1(\Omega) \times L^2(\Omega) \times (0, s_0)$.

Let us write now the *Hamilton–Jacobi–Bellmann (HJB) equation*; we take

$$v(x, t) = w(x) \text{ in } \mathcal{O} \times (s, s + \varepsilon). \tag{6.222}$$

With this choice of v, $\{y(t), y_t(t)\}$ 'moves' during the time interval $(s, s+\varepsilon)$ from $\{h^0, h^1\}$ to

$$\{h^0 + \varepsilon h^1, h^1 + \varepsilon w \chi_{\mathcal{O}} - \varepsilon A h^0\} + 0(\varepsilon^2)$$

(assuming that $h^0 \in H_0^1(\Omega) \cap H^2(\Omega)$). Then, according to the optimality principle, we have

$$\begin{aligned}\phi(h^0, h^1, s) \;\; = \;\; & \inf_w \left[\frac{\varepsilon}{2} \int_{\mathcal{O}} w^2 \, dx + \phi(h^0 + \varepsilon h^1, h^1 + \varepsilon w \chi_{\mathcal{O}} \right. \\ & \left. -\varepsilon A h^0, s + \varepsilon)\right] + 0(\varepsilon^2). \end{aligned} \tag{6.223}$$

Expanding ϕ we obtain

$$\frac{\partial \phi}{\partial s} + \left(\frac{\partial \phi}{\partial h^0}, h^1\right) - \left(\frac{\partial \phi}{\partial h^1}, A h^0\right) + \inf_{w \in L^2(\mathcal{O})} \left[\frac{1}{2} \int_{\mathcal{O}} w^2 \, dx + \left(\frac{\partial \phi}{\partial h^1}, w \chi_{\mathcal{O}}\right)\right] = 0. \tag{6.224}$$

This is the *HJB equation.* We have the 'final' condition

$$\phi(h^0, h^1, s_0) = \begin{cases} 0 \text{ if } \{h^0, h^1\} \in E, \\ +\infty \text{ otherwise} \end{cases} \tag{6.225}$$

where E is the set described by $y(s_0; v), y_t(s_0; v)$ when y satisfies $y_{tt} + Ay = v\chi_{\mathcal{O}}$ in $\Omega \times (s_0, T)$, $v \in L^2(\mathcal{O} \times (s_0, T))$ $y = 0$ on $\Gamma \times (s_0, T)$, and (6.221) holds true. This definition is not constructive. See Remark 6.31 below.

Remark 6.30 We emphasize once more that the above approach is fairly *formal.*

Remark 6.31 The *real time optimal policy* is given at time $t \in (0, s_0)$ by

$$u(t) = -\frac{\partial \phi}{\partial h^1}(h^0, h^1, t)\chi_{\mathcal{O}}. \tag{6.226}$$

How to proceed for $t \in (s_0, T)$ seems to be an open question, even from a conceptual point of view.

6.13. On the application of controllability methods to the solution of the Helmholtz equation at large wave numbers

6.13.1. Introduction

Stealth technologies have enjoyed a considerable growth of interest during this last decade both for aircraft and space applications. Due to the *very high frequencies* used by modern radars the computation of the *Radar Cross Section (RCS)* of a full aircraft using the *Maxwell equations* is still a *great challenge* (see Talflove (1992)). From the fact that *boundary integral methods* are not well suited to general coated materials, *field approaches* seem to provide an alternative which is worth exploring.

In this section, we consider a particular application of *controllability methods* to the solution of the *Helmholtz equations* obtained when looking for the *monochromatic solutions of linear wave problems.* The idea here is to go back to the original wave equation and to apply techniques, inspired by controllability, which find its *time periodic* solutions. Indeed, this method (introduced in Bristeau, Glowinski and Périaux (1993a,b)) is in competition with – and is related to – the one in which the wave equation is integrated from 0 to $+\infty$ in order to obtain asymptotically a time periodic solution; it is well known from Lax and Phillips (1989) that if the scattering body is convex then the solution will converge *exponentially* to the periodic solution. On the other hand, for *non-convex* reflectors (which is quite a common situation) the convergence can be very slow; the method described in this section substantially improves the speed of convergence of the asymptotic one, particularly for stiff problems where internal rays can be trapped by successive reflections.

6.13.2. The Helmholtz equation and its equivalent wave problem

Let us consider a scattering body B, of boundary $\partial B = \gamma$, 'illuminated' by an *incident monochromatic wave* of frequency $f = k/2\pi$ (see Figure 60).

In the case of the wave equation $u_{tt} - \Delta u = 0$ with a periodic solution $u = \mathrm{Re}\,(U\mathrm{e}^{-ikt})$, the associated *Helmholtz equation*, satisfied by the coefficient $U(x)$ of e^{-ikt} is given by

$$\Delta U + k^2 U = 0 \text{ in } \mathbb{R}^d \backslash \bar{B} (d = 2, 3), \tag{6.227}$$

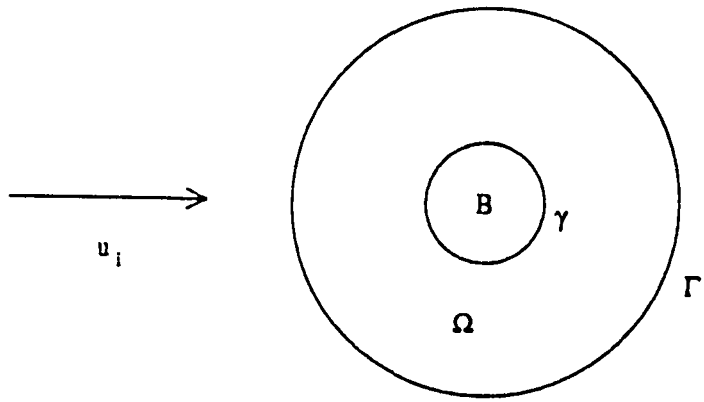


Fig. 60. u_i is the incident field.

$$U = G \text{ on } \gamma. \tag{6.228}$$

In practice, we bound $\mathbb{R}^d \backslash \bar{B}$ by an artificial boundary Γ on which we prescribe, for example, an *approximate first-order Sommerfeld condition such as*

$$\frac{\partial U}{\partial n} - \mathrm{i}kU = 0 \text{ on } \Gamma; \tag{6.229}$$

now, equation (6.227) is prescribed on Ω only, where Ω is this portion of $\mathbb{R}^d \backslash \bar{B}$ between γ and Γ. In the above equations, U is the *scattered field*, $-G$ is the incident field, U and G are *complex valued functions.*

Remark 6.32 More complicated (and efficient) absorbing conditions than (6.229) have been coupled to the controllability method described hereafter; they allow the use of smaller computational domains. The resulting methodology will be described in a forthcoming publication.

Systems (6.227)–(6.229) is related to the T-periodic solutions ($T = 2\pi/k$) of the following wave equation and associated boundary conditions

$$u_{tt} - \Delta u = 0 \text{ in } Q(= \Omega \times (0, T)), \tag{6.230}$$

$$u = g \text{ on } \sigma(= \gamma \times (0, T)), \tag{6.231}$$

$$\frac{\partial u}{\partial n} + \frac{\partial u}{\partial t} = 0 \text{ on } \Sigma(= \Gamma \times (0, T)), \tag{6.232}$$

where, in (6.231), $g(x, t)$ is a time periodic function related to G by $g(x, t) = \operatorname{Re}(\mathrm{e}^{-\mathrm{i}kt} G(x))$. If we denote

$$G(x) = G_{\mathrm{r}}(x) + \mathrm{i}G_{\mathrm{im}}(x),$$

g satisfies

$$g(x, t) = G_{\mathrm{r}}(x) \cos kt + G_{\mathrm{im}}(x) \sin kt.$$

The goal, here, is to find *periodic solutions* to system (6.230)–(6.232) without solving the Helmholtz problem (6.227)–(6.229).

In the following, we look for T-periodic solutions to systems such as (6.230)–(6.232); this means solutions satisfying

$$u(0) = u(T), \quad u_t(0) = u_t(T). \tag{6.233}$$

These solutions can be written

$$u(x,t) = \mathrm{Re}\,(\mathrm{e}^{-\mathrm{i}kt} U(x))$$

(or $u(x,t) = U_{\mathrm{r}} \cos kt + U_{\mathrm{im}} \sin kt$) where $U = U_{\mathrm{r}} + \mathrm{i}U_{\mathrm{im}}$ is the solution of (6.227)–(6.228); so we have

$$u(0) = U_{\mathrm{r}}, u_t(0) = kU_{\mathrm{im}}.$$

6.13.3. Exact controllability methods for the calculation of time periodic solutions to the wave equation

In order to solve problem (6.230)–(6.233) we advocate the following approach whose main merit is to reduce the above problem to an *exact controllability* one, close to those problems whose solution is discussed in Sections 6.1 to 6.12. Indeed, problem (6.230)–(6.233) is clearly equivalent to the following one:

Find $\mathbf{e} = \{e^0, e^1\}$ *such that*

$$u_{tt} - \Delta u = 0 \text{ in } Q, \tag{6.234}$$

$$u = g \text{ on } \sigma, \tag{6.235}$$

$$\frac{\partial u}{\partial n} + \frac{\partial u}{\partial t} = 0 \text{ on } \Sigma, \tag{6.236}$$

$$u(0) = e^0, \quad u_t(0) = e^1, \quad u(T) = e^0, \quad u_t(T) = e^1. \tag{6.237}$$

Problem (6.234)–(6.237) is an *exact controllability problem* which can be solved by methods directly inspired by those in Sections 6.1 to 6.10. We shall not address here the *existence* and *uniqueness* of solutions to problem (6.234)–(6.237) (these issues are addressed in Bardos and Rauch (1994)); instead we shall focus on the practical calculation of such solutions, assuming they do exist.

6.13.4. Least-squares formulation of the problem (6.234)–(6.237)

In order to be able to apply controllability methods to the solution of problem (6.234)–(6.237) the appropriate choice for the space E containing $\mathbf{e} = \{e^0, e^1\}$ is fundamental. We advocate

$$E = V_g \times L^2(\Omega), \tag{6.238}$$

where

$$V_g = \{\varphi \mid \varphi \in H^1(\Omega), \varphi = g(0) \text{ on } \gamma\}. \tag{6.239}$$

In order to solve (6.234)–(6.237), we use the following *least-squares* formulation (where y plays the role of u in (6.234)–(6.237)):

$$\min_{\mathbf{v}\in E} J(\mathbf{v}) \tag{6.240}$$

with

$$J(\mathbf{v}) = \frac{1}{2}\int_\Omega (|\nabla(y(T) - v^0)|^2 + |y_t(T) - v^1|^2)\,\mathrm{d}x, \ \forall \mathbf{v} = \{v^0, v^1\}, \tag{6.241}$$

where, in (3.241), the function y is the solution of

$$y_{tt} - \Delta y = 0 \text{ in } Q, \tag{6.242}$$

$$y = g \text{ on } \sigma, \tag{6.243}$$

$$\frac{\partial y}{\partial n} + \frac{\partial y}{\partial t} = 0 \text{ on } \Sigma, \tag{6.244}$$

$$y(0) = v^0, \quad y_t(0) = v^1. \tag{6.245}$$

The choice of J is directly related to the fact that the *natural energy* $\mathcal{E}(\cdot)$ associated with the system is defined by

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega (|\nabla y|^2 + |y_t|^2)\,\mathrm{d}x. \tag{6.246}$$

Assuming that $\mathbf{e}$ is the solution of problem (6.240), it will satisfy the following equation

$$\langle J'(\mathbf{e}), \mathbf{v}\rangle = 0, \ \forall \mathbf{v} \in E_0, \tag{6.247}$$

where, in (6.247), $E_0 = V_0 \times L^2(\Omega)$ (with $V_0 = \{\varphi \mid \varphi \in H^1(\Omega), \varphi = 0 \text{ on } \gamma\}$) and where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E_0' and E_0 (E_0': dual space of E_0). In (6.247), J' denotes the *differential* of J.

Problem (6.247) can be solved by a *conjugate gradient algorithm* (described in Section 6.13.6) operating in E; in order to implement this algorithm, we need to be able to compute $J'(\mathbf{v}), \forall \mathbf{v} \in E$; this is the object of the following section.

6.13.5. Calculation of J'

To compute J' we use a *perturbation analysis*: starting from (6.241), we obtain

$$\begin{aligned} \delta J(\mathbf{v}) &= \langle J'(\mathbf{v}), \delta\mathbf{v}\rangle \\ &= \int_\Omega \nabla(v^0 - y(T)) \cdot \nabla \delta v^0\,\mathrm{d}x + \int_\Omega (v^1 - y_t(T))\delta v^1\,\mathrm{d}x \end{aligned}$$

$$+\int_\Omega \nabla(y(T)-v^0)\cdot\nabla\delta y(T)\,\mathrm{d}x$$
$$+\int_\Omega (y_t(T)-v^1)\delta y_t(T)\,\mathrm{d}x. \tag{6.248}$$

We also have from (6.242)–(6.245):

$$\delta y_{tt}-\Delta\delta y=0 \text{ in } Q, \tag{6.249}$$

$$\delta y=0 \text{ on } \sigma, \tag{6.250}$$

$$\left(\frac{\partial}{\partial n}+\frac{\partial}{\partial t}\right)\delta y=0 \text{ on } \Sigma, \tag{6.251}$$

$$\delta y(0)=\delta v^0, \quad \delta y_t(0)=\delta v^1. \tag{6.252}$$

Consider now a function p of x and t such that the function $p(t): x\to p(x,t)$ vanishes on γ; next, multiply both sides of (6.249) by p, integrate on Q and then by parts. It follows then from (6.250), (6.251) that

$$\int_\Omega \delta y_t p\,\mathrm{d}x|_0^T-\int_\Omega \delta y p_t\,\mathrm{d}x|_0^T+\int_Q p_{tt}\delta y\,\mathrm{d}x\,\mathrm{d}t+\int_Q \nabla p\cdot\nabla\delta y\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_\Gamma \delta y p\,\mathrm{d}\Gamma|_0^T-\int_\Sigma p_t\delta y\,\mathrm{d}\Gamma\,\mathrm{d}t=0. \tag{6.253}$$

Suppose that p satisfies

$$\int_\Omega (p_{tt}z+\nabla p\cdot\nabla z)\,\mathrm{d}x-\int_\Gamma p_t z\,\mathrm{d}\Gamma=0,\ \forall z\in V_0, p=0 \text{ on } \sigma, \text{ a.e. on } (0,T); \tag{6.254}$$

(6.254) is *equivalent* to

$$p_{tt}-\Delta p=0 \text{ in } Q, \tag{6.255}$$

$$p=0 \text{ on } \sigma, \tag{6.256}$$

$$\frac{\partial p}{\partial n}-\frac{\partial p}{\partial t}=0 \text{ on } \Sigma. \tag{6.257}$$

Using (6.252), equation (6.253) reduces then to

$$\int_\Omega \delta y_t(T)p(T)\,\mathrm{d}x-\int_\Omega \delta y(T)p_t(T)\,\mathrm{d}x+\int_\Gamma \delta y(T)p(T)\,\mathrm{d}\Gamma'$$
$$=\int_\Omega \delta y_t(0)p(0)\,\mathrm{d}x-\int_\Omega \delta y(0)p_t(0)\,\mathrm{d}x+\int_\Gamma \delta y(0)p(0)\,\mathrm{d}\Gamma$$
$$=\int_\Omega p(0)\delta v^1\,\mathrm{d}x-\int_\Omega p_t(0)\delta v^0\,\mathrm{d}x+\int_\Gamma p(0)\delta v^0\,\mathrm{d}\Gamma. \tag{6.258}$$

Let us define $p(T)$ and $p_t(T)$ by

$$p(T)=y_t(T)-v^1 \tag{6.259}$$

and

$$\int_\Omega p_t(T) z \, dx = \int_\Gamma (y_t(T) - v^1) z \, d\Gamma - \int_\Omega \nabla(y(T) - v^0) \cdot \nabla z \, dx, \ \forall z \in V_0, \tag{6.260}$$

respectively. Finally, using (6.248) and (6.258)–(6.260), with $z = \delta y(T)$, shows that

$$\begin{aligned} \langle J'(\mathbf{v}), \mathbf{w} \rangle &= \int_\Omega \nabla(v^0 - y(T)) \cdot \nabla w^0 \, dx - \int_\Omega p_t(0) w^0 \, dx \\ &\quad + \int_\Gamma p(0) w^0 \, d\Gamma + \int_\Omega p(0) w^1 \, dx + \int_\Omega (v^1 - y_t(T)) w^1 \, dx, \\ \forall \mathbf{w} &= \{w^0, w^1\} \in E_0, \end{aligned} \tag{6.261}$$

where, in (6.261), p is the solution of the *adjoint equation* (6.255)–(6.257), completed by the 'final conditions' (6.259), (6.260).

Remark 6.33 Relations (6.260) and (6.261) are largely formal; however, it is worth mentioning that the discrete variants of these two relations make sense and lead to algorithms with fast convergence properties.

Remark 6.34 The *well-posedness* of problem (6.240) is discussed in Bardos and Rauch (1994), where sufficient conditions for existence and uniqueness are given.

6.13.6. Conjugate gradient solution of the least-squares problem (6.240)

A *conjugate gradient algorithm* for the solution of the *linear problem* (6.247) (equivalent to problem (6.240)) is given by

Step 0: Initialization

$$\mathbf{e}_0 = \{e_0^0, e_0^1\} \text{ is given in } E. \tag{6.262}$$

Solve the following forward wave problem

$$\frac{\partial^2 y_0}{\partial t^2} - \Delta y_0 = 0 \text{ in } Q, \tag*{(6.263)$_1$}$$

$$y_0 = g \text{ on } \sigma, \tag*{(6.263)$_2$}$$

$$\frac{\partial y_0}{\partial n} + \frac{\partial y_0}{\partial t} = 0 \text{ on } \Sigma, \tag*{(6.263)$_3$}$$

$$y_0(0) = e_0^0, \ \frac{\partial y_0}{\partial t}(0) = e_0^1. \tag*{(6.263)$_4$}$$

Solve the following backward wave problem

$$\frac{\partial^2 p_0}{\partial t^2} - \Delta p_0 = 0 \text{ in } Q, \tag*{(6.264)$_1$}$$

$$p_0 = 0 \text{ on } \sigma, \tag*{(6.264)$_2$}$$

$$\frac{\partial p_0}{\partial n} - \frac{\partial p_0}{\partial t} = 0 \text{ on } \Sigma, \tag{6.264$_3$}$$

with $p_0(T)$ *and* $(\partial p_0/\partial t)(T)$ *given by*

$$p_0(T) = \frac{\partial y_0}{\partial t}(T) - e_0^1, \tag{6.264$_4$}$$

$$\int_\Omega \frac{\partial p_0}{\partial t}(T) z \, \mathrm{d}x = \int_\Gamma p_0(T) z \, \mathrm{d}\Gamma - \int_\Omega \boldsymbol{\nabla}(y_0(T) - e_0^0) \cdot \boldsymbol{\nabla} z \, \mathrm{d}x, \ \forall z \in V_0, \tag{6.264$_5$}$$

respectively.

Define next $\mathbf{g}_0 = \{g_0^0, g_0^1\} \in E_0 (= V_0 \times L^2(\Omega))$ *by*

$$\begin{aligned}\int_\Omega \boldsymbol{\nabla} g_0^0 \cdot \boldsymbol{\nabla} z \, \mathrm{d}x \ = \ & \int_\Omega \boldsymbol{\nabla}(e_0^0 - y_0(T)) \cdot \boldsymbol{\nabla} z \, \mathrm{d}x - \int_\Omega \frac{\partial p_0}{\partial t}(0) z \, \mathrm{d}x \\ & + \int_\Gamma p_0(0) z \, \mathrm{d}\Gamma, \ \forall z \in V_0,\end{aligned} \tag{6.265$_1$}$$

$$g_0^1 = p_0(0) + e_0^1 - \frac{\partial y_0}{\partial t}(T), \tag{6.265$_2$}$$

and then

$$\mathbf{w}^0 = \mathbf{g}^0. \tag{6.266}$$

For $k \geq 0$, *suppose that* $\mathbf{e}_k$, $\mathbf{g}_k$, $\mathbf{w}_k$ *are known, we compute then* $\mathbf{e}_{k+1}$, $\mathbf{g}_{k+1}$, $\mathbf{w}_{k+1}$ *as follows*

Step 1: Descent

$$\frac{\partial^2 \bar{y}_k}{\partial t^2} - \Delta \bar{y}_k = 0 \text{ in } Q, \tag{6.267$_1$}$$

$$\bar{y}_k = 0 \text{ on } \sigma, \tag{6.267$_2$}$$

$$\frac{\partial \bar{y}_k}{\partial t} + \frac{\partial \bar{y}_k}{\partial n} = 0 \text{ on } \Sigma, \tag{6.267$_3$}$$

$$\bar{y}_k(0) = w_k^0, \quad \frac{\partial \bar{y}_k}{\partial t}(0) = w_k^1. \tag{6.267$_4$}$$

Solve then

$$\frac{\partial^2 \bar{p}_k}{\partial t} - \Delta \bar{p}_k = 0 \text{ in } Q, \tag{6.268$_1$}$$

$$\bar{p}_k = 0 \text{ on } \sigma, \tag{6.268$_2$}$$

$$\frac{\partial \bar{p}_k}{\partial t} - \frac{\partial \bar{p}_k}{\partial n} = 0 \text{ on } \Sigma, \tag{6.268$_3$}$$

with $\bar{p}_k(T)$ *and* $(\partial \bar{p}_k/\partial t)(T)$ *given by*

$$\bar{p}_k(T) = \frac{\partial \bar{y}_k}{\partial t}(T) - w_k^1, \tag{6.268$_4$}$$

$$\int_\Omega \frac{\partial \bar{p}_k}{\partial t}(T) z \,\mathrm{d}x = \int_\Gamma \bar{p}_k(T) z \,\mathrm{d}\Gamma - \int_\Omega \boldsymbol{\nabla}(\bar{y}_k(T) - w_k^0) \cdot \boldsymbol{\nabla} z \,\mathrm{d}x, \ \forall z \in V_0, \tag{6.268$_5$}$$

respectively. Define next $\bar{\mathbf{g}}_k = \{\bar{g}_k^0, \bar{g}_k^1\} \in V_0 \times L^2(\Omega)$ *by*

$$\begin{aligned}\int_\Omega \boldsymbol{\nabla}\bar{g}_k^0 \cdot \boldsymbol{\nabla} z \,\mathrm{d}x \ = \ & \int_\Omega \boldsymbol{\nabla}(w_k^0 - \bar{y}_k(T)) \cdot \boldsymbol{\nabla} z \,\mathrm{d}x - \int_\Omega \frac{\partial \bar{p}_k}{\partial t}(0) z \,\mathrm{d}x \\ & + \int_\Gamma \bar{p}_k(0) z \,\mathrm{d}\Gamma, \ \forall z \in V_0,\end{aligned} \tag{6.269$_1$}$$

$$\bar{g}_k^1 = \bar{p}_k(0) + w_k^1 - \frac{\partial \bar{y}_k}{\partial t}(T), \tag{6.269$_2$}$$

and then ρ_k *by*

$$\rho_k = \int_\Omega (|\boldsymbol{\nabla} g_k^0|^2 + |g_k^1|^2)\,\mathrm{d}x \Big/ \int_\Omega (\boldsymbol{\nabla}\bar{g}_k^0 \cdot \boldsymbol{\nabla} w_k^0 + \bar{g}_k^1 w_k^1)\,\mathrm{d}x. \tag{6.270}$$

We update then $\mathbf{e}_k$ *and* $\mathbf{g}_k$ *by*

$$\mathbf{e}_{k+1} = \mathbf{e}_k - \rho_k \mathbf{w}_k, \tag{6.271}$$

$$\mathbf{g}_{k+1} = \mathbf{g}_k - \rho_k \bar{\mathbf{g}}_k. \tag{6.272}$$

Step 2: Test of the convergence and construction of the new descent direction. If $(\int_\Omega (|\boldsymbol{\nabla} g_{k+1}^0|^2 + |g_{k+1}^1|^2)\,\mathrm{d}x)^{1/2} / (\int_\Omega (|\boldsymbol{\nabla} g_0^0|^2 + |g_0^1|^2)\,\mathrm{d}x)^{1/2} \leq \epsilon$ *take* $\mathbf{e} = \mathbf{e}_{k+1}$*; if not, compute*

$$\gamma_k = \int_\Omega (|\boldsymbol{\nabla} g_{k+1}^0|^2 + |g_{k+1}^1|^2)\,\mathrm{d}x \Big/ \int_\Omega (|\boldsymbol{\nabla} g_k^0|^2 + |g_k^1|^2)\,\mathrm{d}x \tag{6.273}$$

and update $\mathbf{w}_k$ *by*

$$\mathbf{w}_{k+1} = \mathbf{g}_{k+1} + \gamma_k \mathbf{w}_k. \tag{6.274}$$

Do $k = k + 1$ *and go to* (6.267).

Remark 6.35 Algorithm (6.262)–(6.274) looks complicated at first glance. In fact, it is not that complicated to implement since each iteration requires basically the solution of *two wave equations* such as (6.267) and (6.268) and of an *elliptic problem* such as $(6.269)_1$. The above problems are classical ones for which efficient solution methods already exist.

Remark 6.36 Algorithm (6.262)–(6.274) can be seen as a variation of the *asymptotic method* mentioned in Section 6.13.1; there, we integrate the periodically excited wave equation until we reach a periodic solution (i.e. a *limit cycle*). What algorithm (6.262)–(6.274) does indeed is to *periodically* measure the *lack* (or *defect) of periodicity* and use the result of this measure as a *residual* to speed up the convergence to a periodic solution. In fact, a similar idea was used in Auchmuty, Dean, Glowinski and Zhang (1987) to compute the periodic solutions of systems of *stiff* nonlinear differential equations (including cases where the period itself is an unknown parameter of the problem).

6.13.7. An FD/FE implementation

The practical implementation of the previously presented control-based method is straightforward. It is based on a *time discretization* by the *centred second-order accurate explicit FD scheme*, already employed in Sections 6.8 and 6.9. This scheme is combined to *piecewise linear FE approximations* (as in Sections 6.8 and 6.9) for the space variables; we use *mass lumping* – through numerical integration by the *trapezoïdal rule* – to obtain a *diagonal* mass matrix for the acceleration term. The fully discrete scheme has to satisfy a *stability condition* such as $\Delta t \leq Ch$, where C is a constant. To obtain accurate solutions, we need to have h at least *ten times smaller* than the wavelength; in fact, h has to be even smaller ($h \approx \lambda/20$) in those regions where internal rays are trapped by successive reflections. For the *mesh generation*, the advancing front method proposed by George (1971) has been used; this method (implemented at INRIA by George and Seveno) gives *homogeneous* triangulations (see the following figures).

6.13.8. Numerical experiments.

In order to validate the methods discussed in the above sections, we have considered the solution of three test problems of increasing difficulty, namely the scattering of planar incident waves by a *disk*, then by a *nonconvex* reflector which can be seen as a semi-open cavity (a kind of – very – idealized air intake) and finally the scattering of a planar wave by a *two-dimensional aircraft related body*. For these different cases the artificial boundary is located at a 3λ distance from B and we assume that the boundary of the reflector is *perfectly conducting.*

The following results have been obtained by Bristeau at INRIA (see Bristeau, Glowinski and Périaux (1993a,b,c) for further numerical experiments and details).

Scattering by a disk. Before discussing our numerical experiments, let us observe that model (6.234)–(6.236) assumes, implicitly, that its solutions propagates with velocity 1, implying that, here, the wavelength is equal to the period. If $c(> 0)$ is different from 1, we shall rescale x and t, so that $c = 1$. In the following examples, the data are given in the MKSA system before rescaling.

Example 1 (Scattering by a disk) For this problem, B is a disk of radius 0.25 m, $k = 2\pi f$ with $f = 2.4$ GHz, implying that the wavelength is 0.125 m; the disk is illuminated by an incident *planar wave* coming from the left. The artificial boundary is located at a distance of 3λ from B. The *FE triangulation* has 18,816 vertices and 36,970 triangles; the mean length of the edges is $\lambda/15$, the minimal value being $\lambda/28$, while the maximal one is $\lambda/10$.

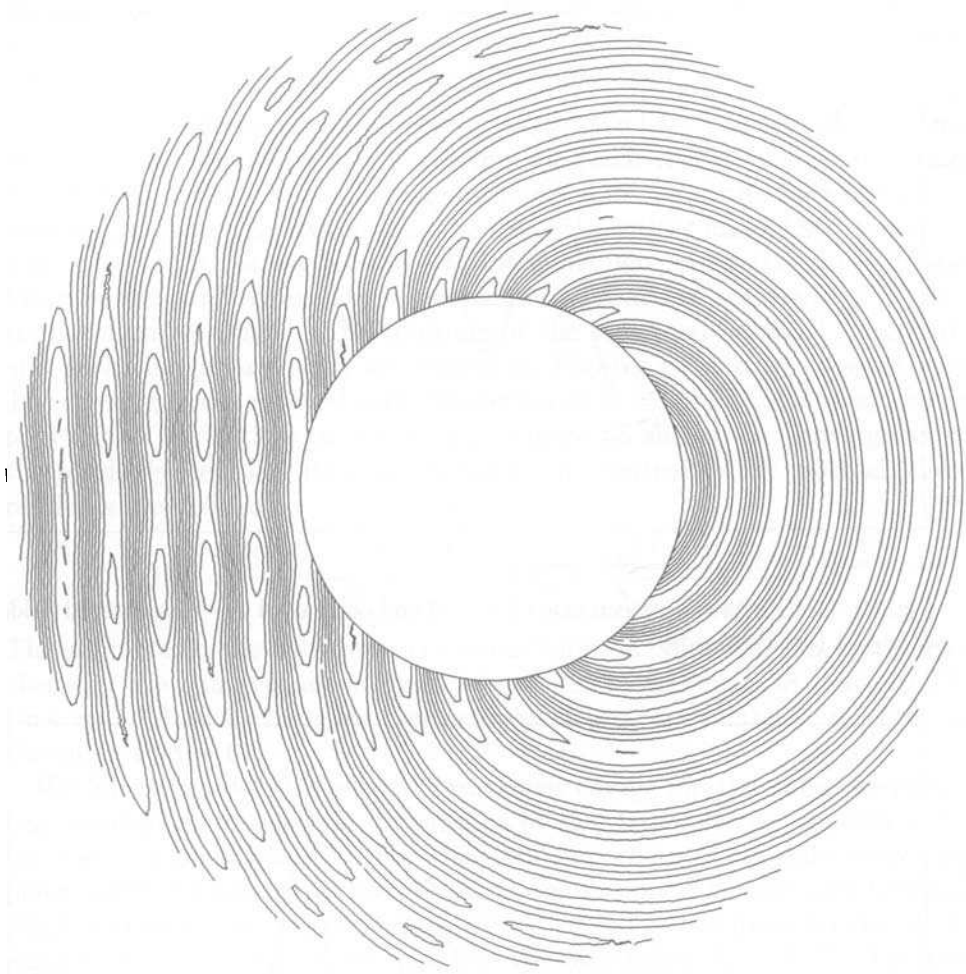


Fig. 61. Contours of the scattered field (real component).

The value of Δt is $T/35$. To obtain convergence of the iterative method, 74 iterations of algorithm (6.262)–(6.274) were needed (with $\epsilon = 5 \times 10^{-5}$ for the stopping criterion) corresponding to a 3 min computation on a CRAY2. Figure 61 shows the scattered field e^0 (*real component* of the Helmholtz problem solution). For this test problem where the exact solution is known, we have compared on Figures 62 and 63 the computed solution (——) with the exact one ($\cdots\cdots$) on two cross sections (incident direction, opposite to incident direction, respectively). Of course, for this problem, the asymptotic method (just integrating the wave equation from 0 to nT, n 'large') is less CPU time consuming; we have chosen this example just to test the accuracy of the approximations.

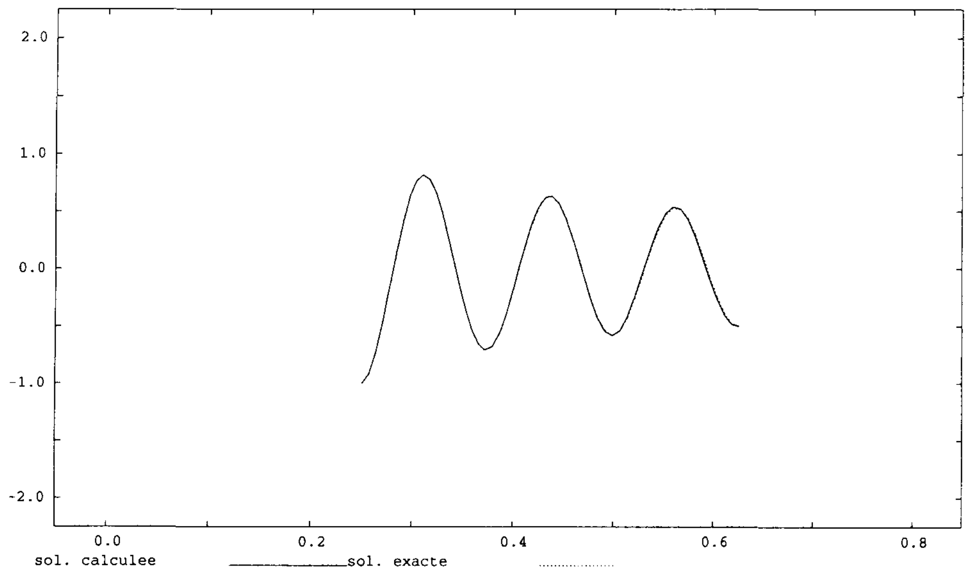


Fig. 62. Comparison between exact (······) and computed (——) scattered fields $e^0(x_1, 0)$ (incident side).

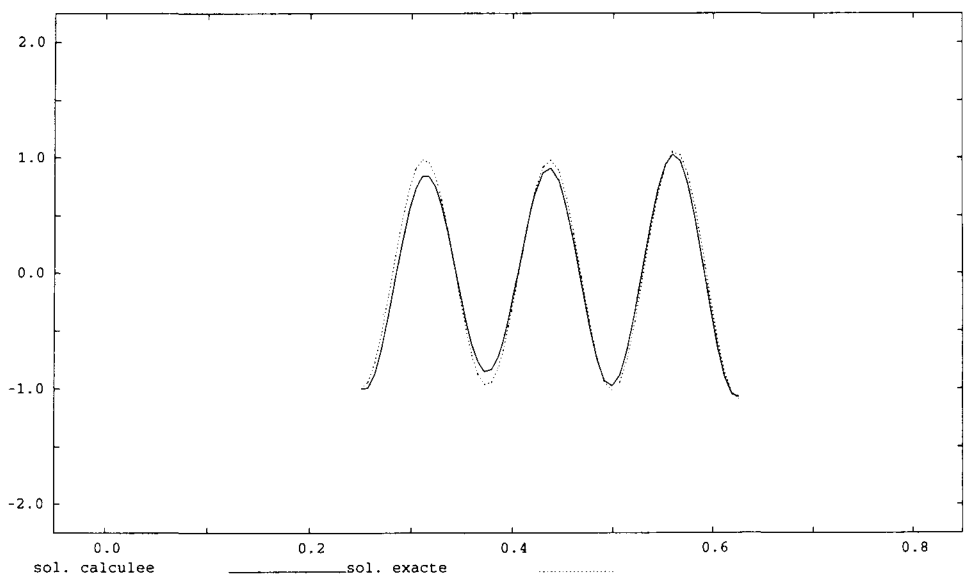


Fig. 63. Comparison between exact (······) and computed (——) scattered fields $e^0(x_1, 0)$ (shadow side).

Remark 6.37 We can substantially increase the accuracy by using on Γ instead of (6.232), *second-order absorbing boundary conditions* like those discussed in Bristeau, Glowinski and Périaux (1993c).

Example 2 (Scattering by semi-open cavities) We have considered two semi-open cavities. We choose $f = 3$ GHz implying that the wavelength is 0.10 m. For the first cavity, the inside dimensions are $4\lambda \times \lambda$ and the thickness of the wall is $\lambda/5$. The FE triangulation has 22,951 vertices and 44,992 triangles. The value of Δt corresponds to 40 time steps per period (i.e. $\Delta t = T/40$). We consider an illuminating monochromatic wave of incidence $\alpha = 30°$, coming from the right. The contours of the scattered fields e^0 (real part) and e^1/k (imaginary part) are shown on Figures 64 and 65, respectively. The convergence is reached with 136 iterations ($\epsilon = 5 \times 10^{-5}$), corresponding to 8 min of CPU time on a CRAY2. Figure 66 shows the convergence of the residuals $\mathrm{Re}\,^0_k$ and $\mathrm{Re}\,^1_k$ associated to the controllability method; these residuals are defined by

$$\mathrm{Re}\,^0_k = \frac{\|e^0_{k+1} - e^0_k\|_{L^2(\Omega)}}{\|e^0_1 - e^0_0\|_{L^2(\Omega)}}, \quad \mathrm{Re}\,^1_k = \frac{\|e^1_{k+1} - e^1_k\|_{L^2(\Omega)}}{\|e^1_1 - e^1_0\|_{L^2(\Omega)}}.$$

The asymptotic method gives the same solution, but, for this *nonconvex* obstacle, the convergence is *much slower* (800 iterations, 18 min of CPU time on a CRAY2) than the convergence of the controllability method, as shown on Figure 67.

We have considered a second semi-open cavity for the same frequency and wavelength; the inside dimensions of this larger cavity are $20\lambda \times 5\lambda$, the wall thickness being λ. For this problem where many reflections take place inside the cavity, we need a fine triangulation. The one used here has 208,015 vertices and 412,028 triangles, with $\lambda/30$ as the mean length of the edges inside the cavity ($\lambda/20$ outside). We have taken $\Delta t = T/70$. The test problem corresponds to an illuminating wave of incidence $\alpha = 30°$, coming from the right. The contours of the total field related to e^0 are shown on Figure 68. Figure 69 shows the convergence of the cost function $J(\mathbf{e}_k)$ with $J(\cdot)$ defined by (6.241); we have also shown on Figure 69 the convergence to zero of the two components of this cost function (the one related to e^0_k, and the one related to e^1_k).

For this difficult case the convergence is slower than for the above cavity problem (300 iterations instead of 136).

We have shown on Figure 70 some details of the FE triangulation close to the wall at the entrance of the cavity.

Example 3 (scattering by a two-dimensional aircraft related body) We consider the reflector defined by the cross section of a Dassault Aviation Falcon 50 by its symmetry plane; the shape of the air-intake is given and we have artificially closed it in order to enhance reflections. The plane length is about

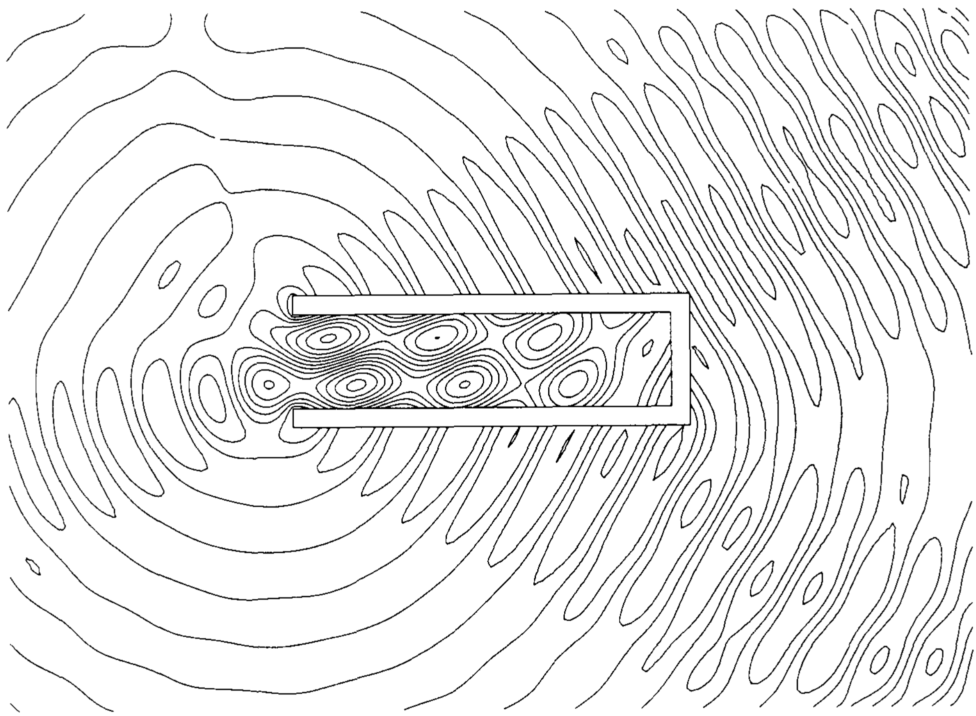


Fig. 64. Contours of the scattered field (real part; $\alpha = 30°$).

18 m, while its height is 6 m. We take $f = 0.6$ GHz, so that $\lambda = 0.5$ m. The FE mesh has 143,850 vertices and 283,873 triangles; Figure 71 shows an enlargement of the mesh near the air intake. We have used $\Delta t = T/40$. We consider an illuminating wave with $\alpha = 0°$ as angle of incidence. The contours of the total field (real part) are presented on Figure 72; we observe the *shadow region* behind the aircraft. The convergence (for $\epsilon = 5 \times 10^{-5}$) is obtained after 260 iterations, corresponding to 90 min of CPU time on a CRAY2; Figure 73 shows the convergence of $J(\mathbf{e}_k)$ to 0 as $k \to +\infty$.

Remark 6.38 For all the test problems discussed above, we have used a direct method based on a *sparse Cholesky solver* to solve the (discrete) elliptic problem encountered at each iteration of the discrete analogue of algorithm (6.262)–(6.274). Despite the respectable size of these systems (up to 2×10^5 unknowns) this part of the algorithm takes no more than a *few percent* of the total computational effort.

Indeed, most of the computational time is spent integrating the forward and backward wave equations; fortunately this is the easiest part to *parallelize* (hopefully in the near future; see Bristeau, Erhel, Glowinski and Périaux (1993)) as it is based on an *explicit time discretization scheme.*

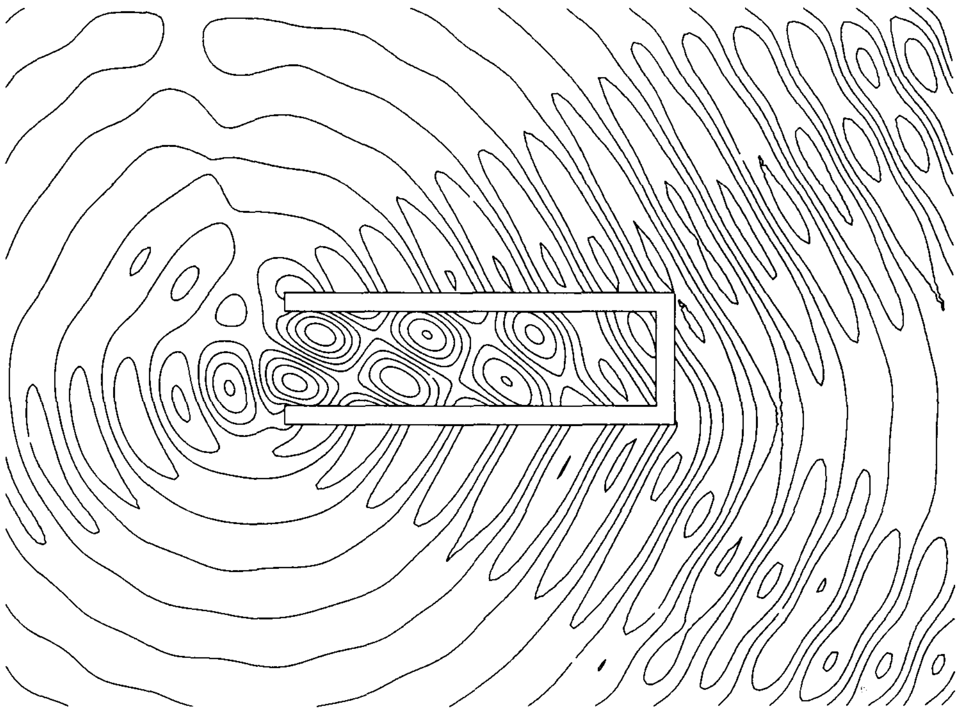


Fig. 65. Contours of the scattered field (imaginary part; $\alpha = 30°$).

6.13.9. Further comments

We have discussed in this section a *controllability* based novel approach to solving the Helmholtz (and two-dimensional harmonic Maxwell) equations for *large wavenumbers* and complicated geometries. The new method so far appears to be more efficient than traditional computational methods which are based on either time asymptotic behaviour or linear algebra algorithms for very large indefinite linear systems.

The new methodology appears to be promising for the *three-dimensional Maxwell equations* and for *heterogeneous media*, including *dissipative* ones. For very large problems, we shall very probably have to combine the above method with *domain decomposition* and/or *fictitious domain methods*, and also to *higher-order* approximations, to reduce the number of grid points.

6.14. Further problems

In this section we have discussed controllability issues concerning wave equations such as

$$u_{tt} - c^2 \Delta u = 0; \tag{6.275}$$

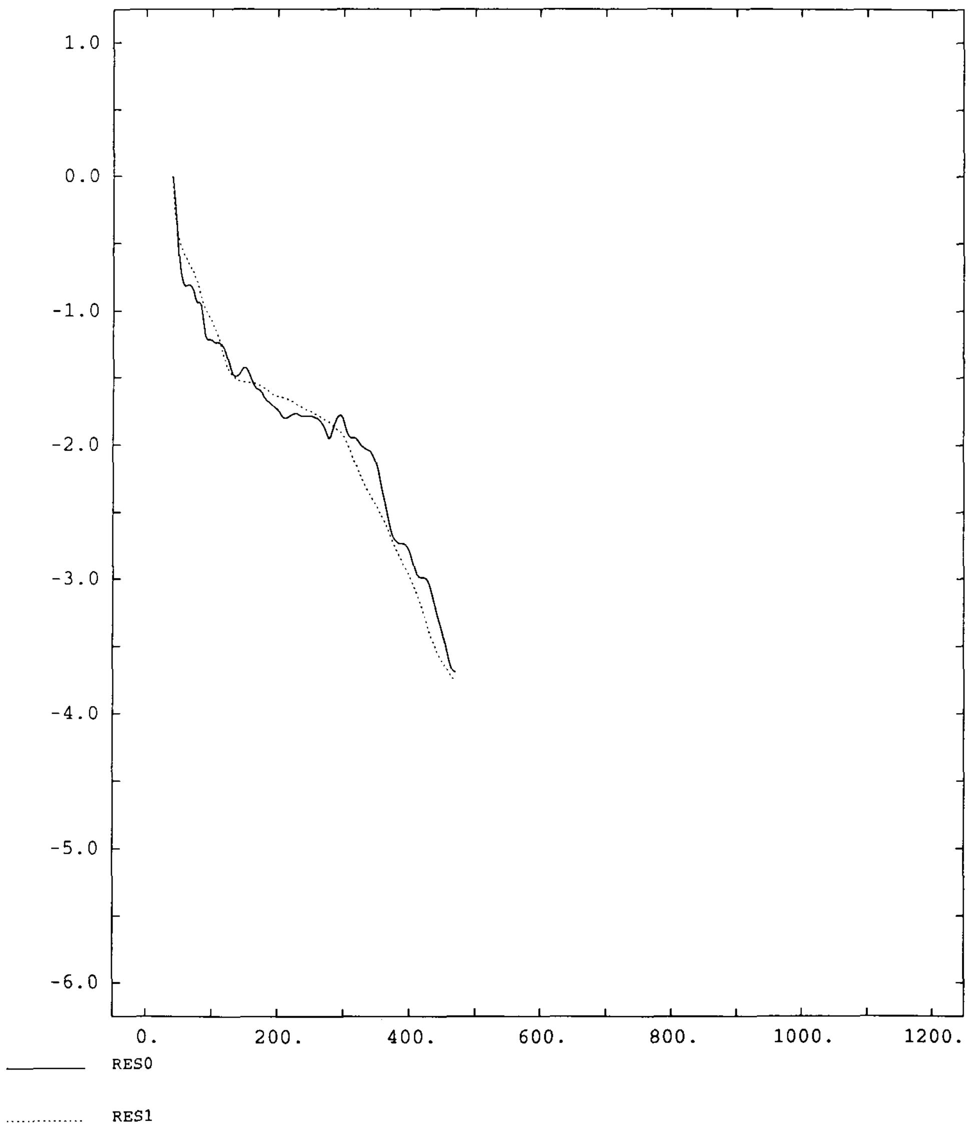


Fig. 66. Convergence of the residual (control solution): ——, residual for y; ······, residual of y_t.

a basic tool for studying exact or approximate controllability for equations such as (3.275) has been the *Hilbert Uniqueness Method* (HUM). Actually, HUM has been applied in Lagnese (1989) to prove the exact boundary controllability of the *Maxwell equations*

$$\varepsilon\frac{\partial \mathbf{E}}{\partial t} - \boldsymbol{\nabla}\times\mathbf{H} = 0, \quad \mu\frac{\partial \mathbf{H}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{E} = 0 \text{ in } Q, \tag{6.276}$$

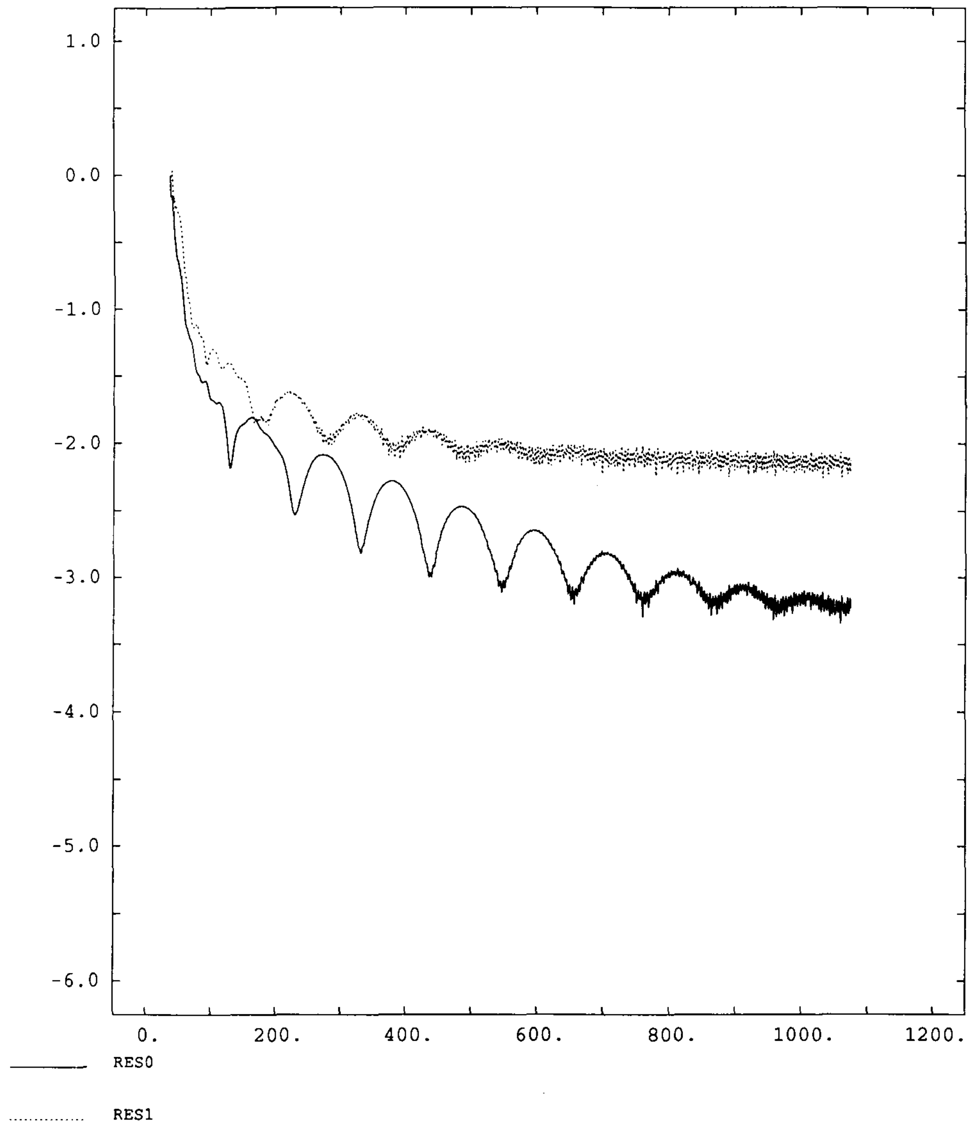


Fig. 67. Convergence of the residual (asymptotic solution): ——, residual for y; $\cdots\cdots$, residual of y_t.

$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{H} = 0 \text{ in } Q \tag{6.277}$$

(see also Bensoussan (1990)); most computational aspects still have to be explored.

The Hilbert Uniqueness Method has been applied in Lions (1988b) and Lagnese and Lions (1988) to the exact or approximate controllability of systems (mostly from *elasticity*) modelled by *Petrowsky's type equations.*

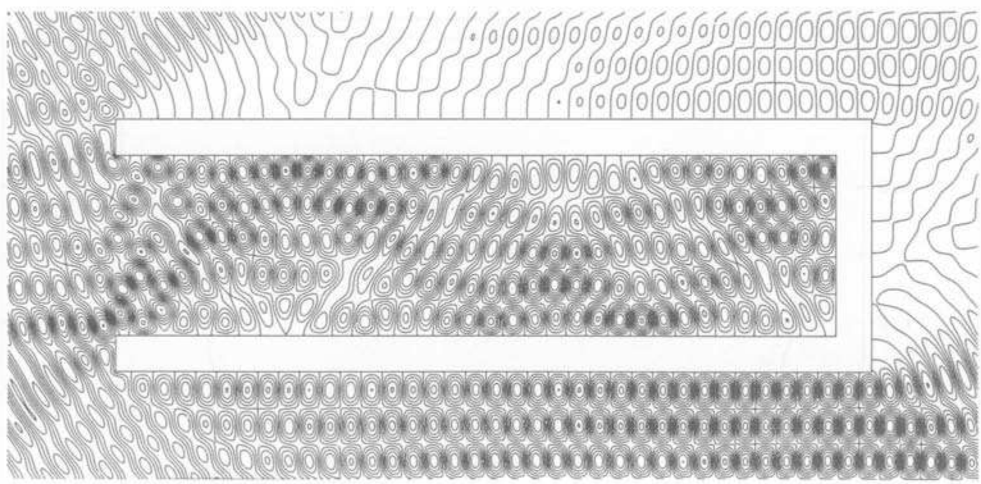


Fig. 68. Contours of the total field ($\alpha = 30°$).

Concerning the numerical application of HUM to the exact controllability of Petrowsky-type equations modelling *elastic shells* vibrations we refer to Marini, Testa and Valente (1994).

Finally, very little is known about the exact or approximate controllability of those (nonlinear) wave (or Petrowsky's type) equations modelling the vibrations of nonlinear systems; we intend, however, to explore the solution of these problems in the near future.

7. COUPLED SYSTEMS

In Sections 1 to 6 we have discussed controllability issues for *diffusion* and *wave* equations, respectively. The control of systems obtained by the coupling of *different types* of equations brings new difficulties which are worth discussing, therefore justifying the present section. The *numerical aspects* will not be addressed here, but in our opinion this Section can be a starting point for investigations in this direction.

In this Section, we shall focus on the controllability of a simplified *Thermoelasticity* system but it is likely that the techniques described here can be applied to systems modelled by more complicated equations.

7.1. A problem from thermoelasticity

Let Ω be a *bounded* domain of $\mathbb{R}^d$, $d \leq 3$, with a smooth boundary Γ. Motivated by applications from *Thermoelasticity* we consider the following

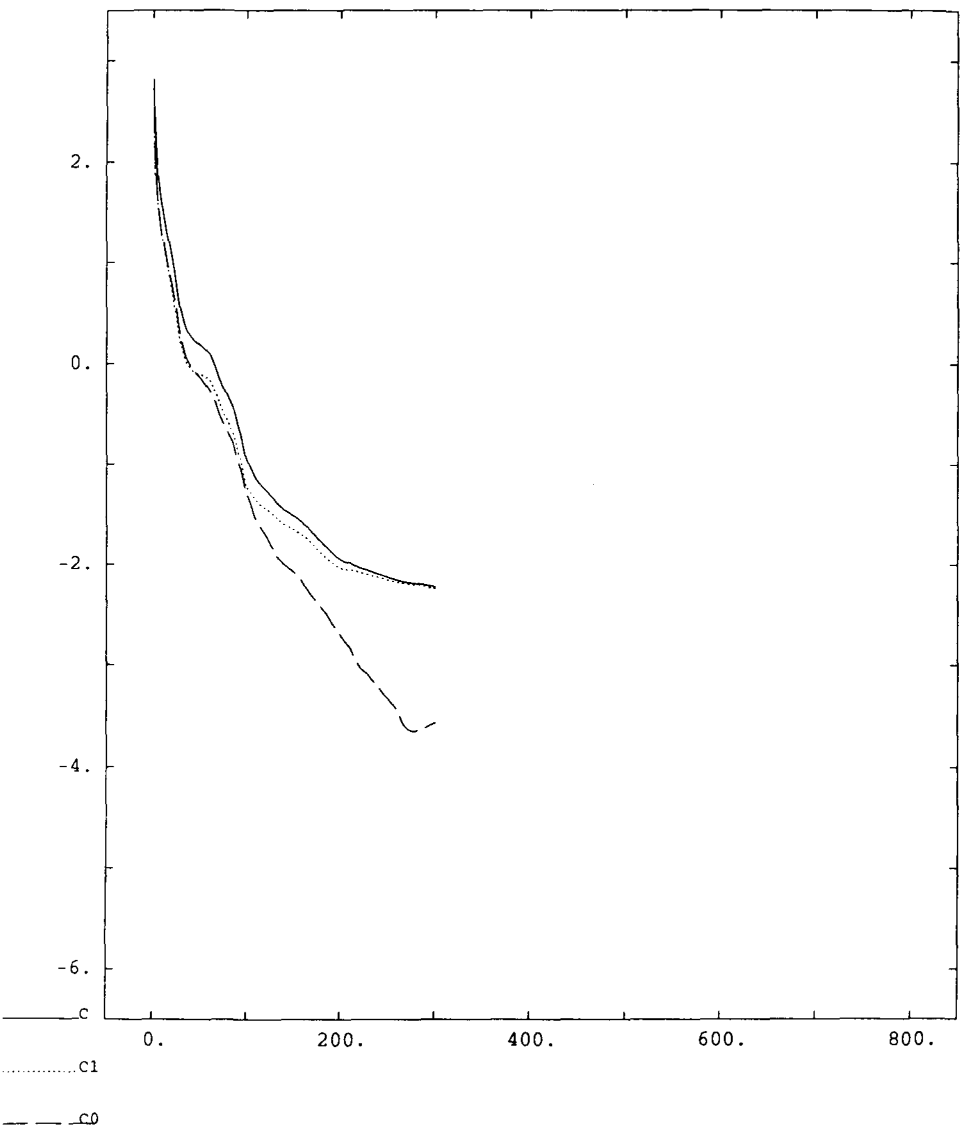


Fig. 69. Convergence of $J(\mathbf{e}_k)$ (——), of the $\mathbf{e}_k^0$ component of $J(\mathbf{e}_k)$ (- - -), and of the $\mathbf{e}_k^1$ component of $J(\mathbf{e}_k)$ (······).

system

$$\frac{\partial^2 \mathbf{y}}{\partial t^2} - \Delta \mathbf{y} + \alpha \nabla \theta = \mathbf{0} \text{ in } Q = \Omega \times (0,T), \qquad (7.1)_1$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta + \alpha \nabla \cdot \frac{\partial \mathbf{y}}{\partial t} = 0 \text{ in } Q, \qquad (7.1)_2$$

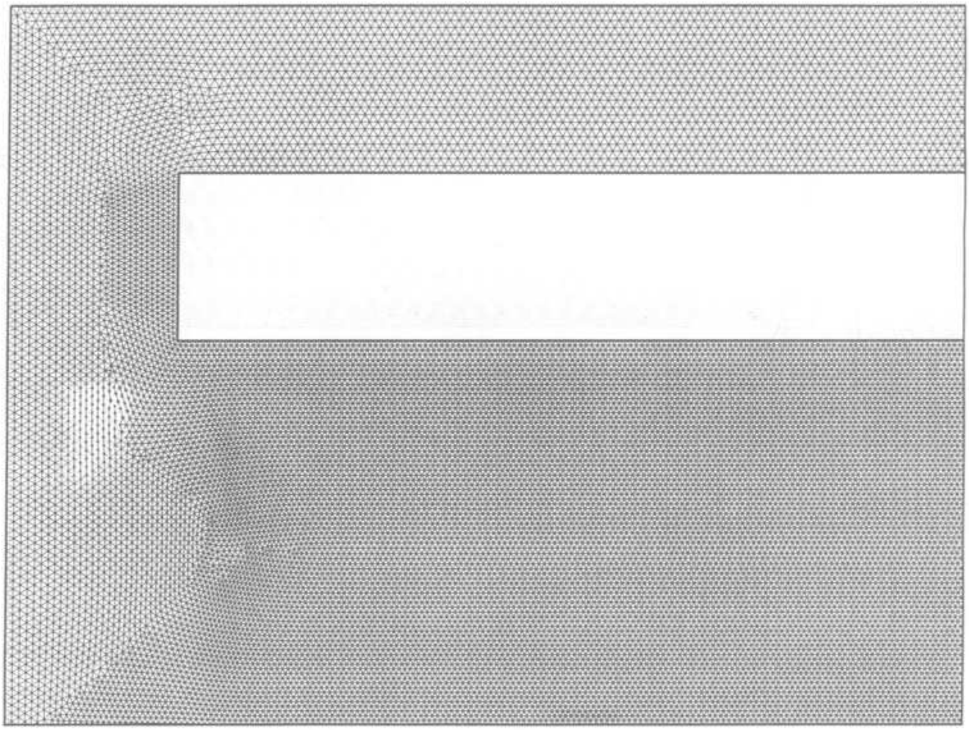


Fig. 70. Enlargement of the mesh close to the cavity intake.

where $\mathbf{y} = \{y_i\}_{i=1}^d, \alpha \geq 0$. In (7.1), $\mathbf{y}$ (respectively θ) denotes an *elastic displacement* (respectively a *temperature*) function of x and t. Scaling has been made so that the constants in front of $-\Delta$ are equal to 1 in both equations.

The *initial conditions* are

$$\mathbf{y}(0) = \mathbf{0}, \quad \frac{\partial \mathbf{y}}{\partial t}(0) = 0, \tag{7.2$_1$}$$

$$\theta(0) = 0. \tag{7.2$_2$}$$

The control is applied on the boundary of Ω, actually on $\Gamma_0 \subset \Gamma$. Also, it is only applied on the component $\mathbf{y}$ of the state vector $\{\mathbf{y}, \theta\}$.

Considering the *boundary conditions*, we shall consider the two following cases:

Case I

$$\mathbf{y} = \begin{cases} \mathbf{v} \text{ on } \Sigma_0 = \Gamma_0 \times (0,T), \mathbf{v} = \{v_i\}_{i=1}^d, \\ \mathbf{0} \text{ on } \Sigma \backslash \Sigma_0, \Sigma = \Gamma \times (0,T) \end{cases} \tag{7.3}$$

and

$$\theta = \theta_0 \text{ is given on } \Sigma. \tag{7.4}$$

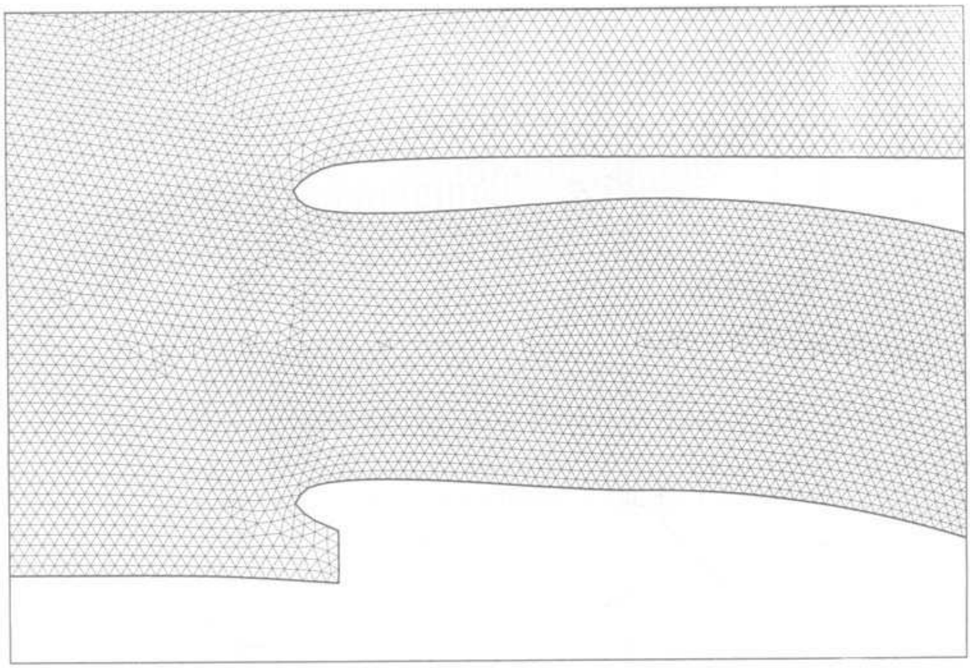


Fig. 71. Enlargement of the mesh close to the aircraft air intake (by courtesy of Dassault Aviation).

Case II

$$\frac{\partial \mathbf{y}}{\partial n} = \begin{cases} \mathbf{v} \text{ on } \Sigma_0, \\ \mathbf{0} \text{ on } \Sigma \backslash \Sigma_0 \end{cases} \tag{7.5}$$

with (7.4) unchanged.

Remark 7.1 One can consider a variety of other types of boundary conditions and controls. The corresponding problems can be treated by methods very close to those given below.

Remark 7.2 In order to simplify the proofs and formulae below, we shall take

$$\theta_0 = 0, \tag{7.6}$$

but this is just a technical detail.

In the following sections, we shall study the spaces described by $\mathbf{y}(T)$, $(\partial \mathbf{y}/\partial t)(T)$ and $\theta(T)$; we shall show that under 'reasonable' conditions, one can control $\mathbf{y}(T)$ and $(\partial \mathbf{y}/\partial t)(T)$ but not $\theta(T)$.

Remark 7.3 Controllability for equations (7.1)–(7.4) has been studied in Lions (1988b,Vol. 2) (see also Narukawa (1983)). We follow here a slightly

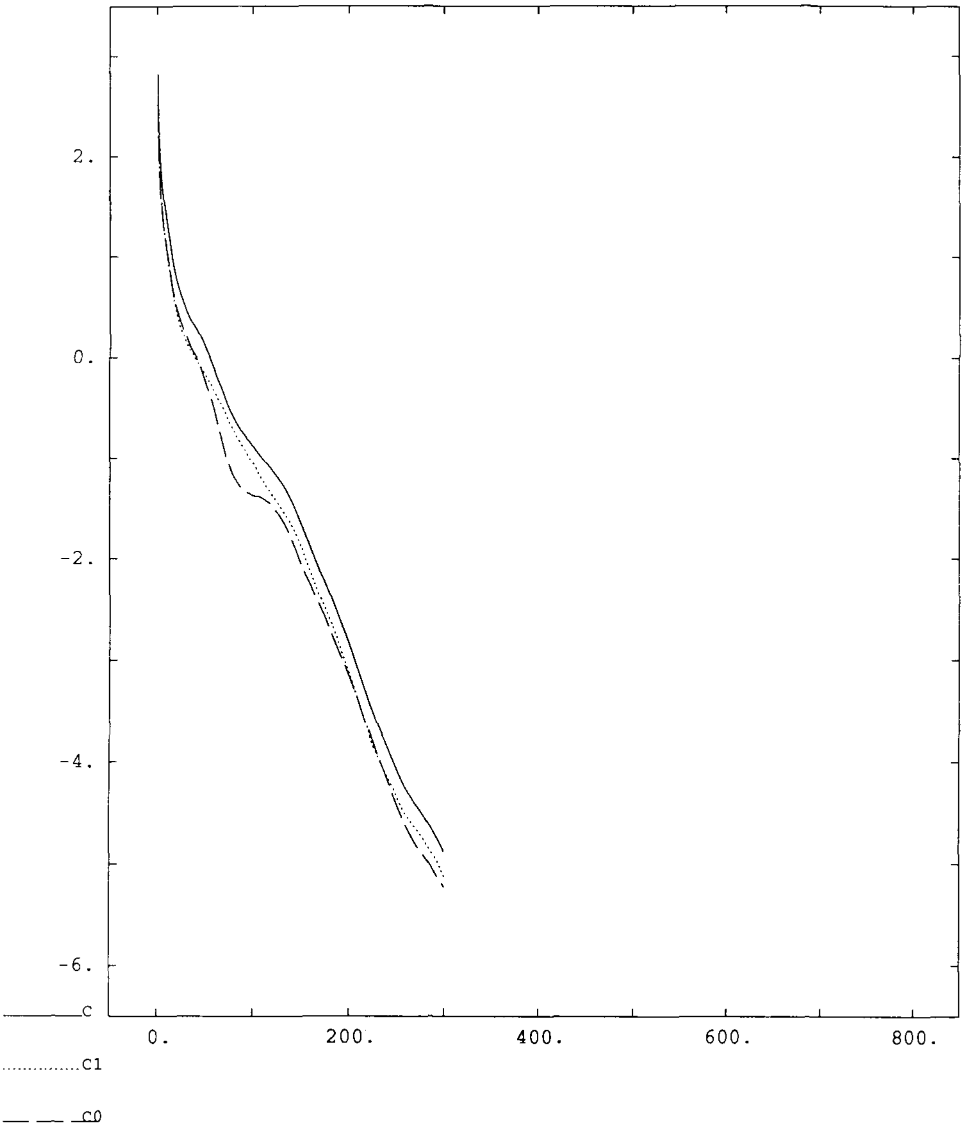


Fig. 72. Convergence of the residuals.

different approach, our goal being to obtain constructive approximation methods.

7.2. The limit cases $\alpha \to 0$ and $\alpha \to +\infty$

In order to obtain a better understanding, the limit cases $\alpha \to 0$ and $\alpha \to +\infty$ are worthwhile looking at. Moreover, they have intrinsic mathematical interest, particularly when $\alpha \to +\infty$.

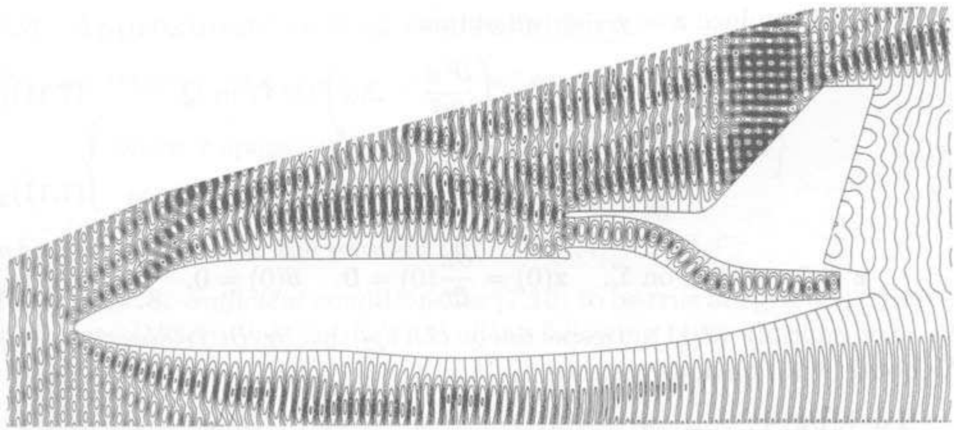


Fig. 73. Contours of the real part of the total field around a Falcon 50 two-dimensional cross section ($\alpha = 0°$) (by courtesy of Dassault Aviation).

7.2.1. The case $\alpha \to 0$.

This case is simple. The coupled system (7.1)–(7.4) (or its variant (7.1), (7.2), (7.4), (7.5)) reduces to *uncoupled* wave and heat equations. The control acts only on the $\mathbf{y}$ components; we then have

$$\frac{\partial^2 \mathbf{y}}{\partial t^2} - \Delta \mathbf{y} = \mathbf{0} \text{ in } Q, \quad \mathbf{y}(0) = \frac{\partial \mathbf{y}}{\partial t}(0) = \mathbf{0}, \ \mathbf{y} = \mathbf{v} \text{ on } \Sigma_0, \ \mathbf{y} = \mathbf{0} \text{ on } \Sigma\backslash\Sigma_0. \tag{7.7}$$

Since Δ is a *diagonal* operator we recover cases discussed in Section 6.

Remark 7.4. The *general linear elasticity* system (with Δ replaced by $\lambda\Delta + \mu graddiv$; λ, μ: Lamé coefficients) would lead to similar considerations, with more complicated technical details.

Remark 7.5. Similar considerations apply when (7.3) is replaced by (7.5).

7.2.2. The case $\alpha \to +\infty$ (boundary conditions (7.3).)

We shall assume (this is *necessary* for what follows) that

$$\int_{\Gamma_0} \mathbf{v} \cdot \mathbf{n} \, d\Gamma = 0. \tag{7.8}$$

Then, assuming $\mathbf{v}$ smooth enough (a condition which does *not* restrict the generality, since we are going to consider *approximate controllability*) and

$$\mathbf{v} = \mathbf{0} \text{ on } \partial\Gamma_0 \times (0,T), \quad \mathbf{v}|_{t=0} = \frac{\partial \mathbf{v}}{\partial t}|_{t=0} = \mathbf{0}, \tag{7.9}$$

one can construct a function ϕ such that

$$\begin{cases} \phi \text{ is smooth in } \bar{\Omega} \times (0,T), \quad \boldsymbol{\nabla} \cdot \phi = 0 \text{ in } \Omega \times (0,T), \\ \phi = \mathbf{v} \text{ on } \Sigma_0, \quad \phi = \mathbf{0} \text{ on } \Sigma\backslash\Sigma_0. \end{cases} \tag{7.10}$$

Then if we introduce $\mathbf{z} = \mathbf{y} - \phi$, we obtain

$$\frac{\partial^2 \mathbf{z}}{\partial t^2} - \Delta \mathbf{z} + \alpha \boldsymbol{\nabla} \theta = -\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta \phi\right) (\equiv \mathbf{f}) \text{ in } Q, \qquad (7.11)_1$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta + \alpha \boldsymbol{\nabla} \cdot \frac{\partial \mathbf{z}}{\partial t} = 0 \text{ in } Q, \qquad (7.11)_2$$

$$\mathbf{z} = \mathbf{0}, \quad \theta = 0 \text{ on } \Sigma, \quad \mathbf{z}(0) = \frac{\partial \mathbf{z}}{\partial t}(0) = \mathbf{0}, \quad \theta(0) = 0. \qquad (7.11)_3$$

We now multiply $(7.11)_1$ (respectively $(7.11)_2$) by $\partial \mathbf{z}/\partial t$ (respectively θ). We obtain with obvious notation ($\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\frac{\partial \mathbf{z}}{\partial t}\|^2 + \|\boldsymbol{\nabla} \mathbf{z}\|^2 + \|\theta\|^2\right] + \|\boldsymbol{\nabla}\theta\|^2 + \alpha\left[\left(\boldsymbol{\nabla}\theta, \frac{\partial \mathbf{z}}{\partial t}\right) + \left(\boldsymbol{\nabla} \cdot \frac{\partial \mathbf{z}}{\partial t}, \theta\right)\right] = \left(\mathbf{f}, \frac{\partial \mathbf{z}}{\partial t}\right). \qquad (7.12)$$

But

$$\left(\boldsymbol{\nabla}\theta, \frac{\partial \mathbf{z}}{\partial t}\right) + \left(\boldsymbol{\nabla} \cdot \frac{\partial \mathbf{z}}{\partial t}, \theta\right) = 0, \qquad (7.13)$$

so that (7.6) leads to *a priori estimates which are independent of* α.

It follows then, that if we denote by $\{\mathbf{z}_\alpha, \theta_\alpha\}$ the solution of (7.11) one has when $\alpha \to +\infty$

$$\begin{cases} \left\{\mathbf{z}_\alpha, \dfrac{\partial \mathbf{z}_\alpha}{\partial t}\right\} \to \left\{\mathbf{z}, \dfrac{\partial \mathbf{z}}{\partial t}\right\} \text{ weakly* in } L^\infty(0,T; H_0^1(\Omega) \times L^2(\Omega)), \\ \theta_\alpha \to \theta \text{ weakly in } L^2(0,T; H_0^1(\Omega)) \text{ and weakly * in } L^\infty(0,T; L^2(\Omega)). \end{cases} \qquad (7.14)$$

Returning to the notation $\{\mathbf{y}, \theta\}$, we have that $\mathbf{y}_\alpha \to \mathbf{y}$, where $\mathbf{y}$ is the solution of

$$\begin{cases} \dfrac{\partial^2 \mathbf{y}}{\partial t^2} - \Delta \mathbf{y} + \boldsymbol{\nabla} p = \mathbf{0} \text{ in } Q, \quad \boldsymbol{\nabla} \cdot \mathbf{y} = 0 \text{ in } Q, \\ \mathbf{y}(0) = \dfrac{\partial \mathbf{y}}{\partial t}(0) = \mathbf{0}, \quad \mathbf{y} = \mathbf{v} \text{ on } \Sigma_0, \mathbf{y} = \mathbf{0} \text{ on } \Sigma \backslash \Sigma_0. \end{cases} \qquad (7.15)$$

We clearly see why (7.8) is necessary (from the *divergence theorem*). We observe that the system satisfied by $\{\mathbf{y}, \theta\}$ is again *uncoupled* at the limit when $\alpha \to +\infty$, so that the best thing we can hope is the controllability of $\{\mathbf{y}(T), (\partial \mathbf{y}/\partial t)(T)\}$, but not, of course, the controllability of $\theta(T)$.

Remark 7.6. A systematic study of the controllability of system (7.15) remains to be done (see, however Lions (1990b) for a discussion of the controllability of system (7.15) under strict geometrical conditions).

Remark 7.7. Similar results hold when (7.3) is replaced by (7.5). Then no additional condition such that (7.8) is needed.

7.3. Approximate partial controllability

We now return to the case $0 < \alpha < +\infty$; we assume that

$$\left\{\begin{array}{l}\text{when } \mathbf{v} \text{ spans } (L^2(\Sigma_0))^d, \text{ then } \left\{\mathbf{y}(T;\mathbf{v}), \dfrac{\partial \mathbf{y}}{\partial t}(T;\mathbf{v})\right\} \\ \text{spans a dense subset of } (L^2(\Omega) \times H^{-1}(\Omega))^d,\end{array}\right. \tag{7.16}$$

where $\{\mathbf{y}(\mathbf{v}), \theta(\mathbf{v})\}$ is the solution of (7.1)–(7.4).

Remark 7.8. *Sufficient* conditions for (7.16) to be true are given in Chapter 1 of Lions (1988b, Vol. 2); they are of the following type:

(i) Σ_0 is 'sufficiently large',
(ii) $0 < \alpha < \alpha_0$.

Necessary and sufficient conditions for (7.16) to be true do not seem to be known. Interesting results have been obtained by E. Zuazua (1993).

We can then consider the following optimal control problem

$$\left\{\begin{array}{l}\inf\limits_{\mathbf{v}} \dfrac{1}{2}\displaystyle\int_{\Sigma_0} |\mathbf{v}|^2 \,\mathrm{d}\Sigma, \quad \mathbf{v} \in (L^2(\Sigma_0))^d \text{ such that} \\ \mathbf{y}(T;\mathbf{v}) \in \mathbf{z}^0 + \beta_0 B, \dfrac{\partial \mathbf{y}}{\partial t}(T;\mathbf{v}) \in \mathbf{z}^1 + \beta_1 B_{-1},\end{array}\right. \tag{7.17}$$

where B (respectively B_{-1}) denotes the unit ball of $(L^2(\Omega))^d$ (respectively of $(H^{-1}(\Omega))^d$).

Problem (7.17) has a unique solution; it can be characterized by a *variational inequality* which can be obtained either directly or by duality methods. Here we use duality, because (among other things) it will be convenient for the next section (where we introduce penalty arguments).

Formulation of a dual problem We follow the same approach as in Section 6.4. We define an operator L from $(L^2(\Sigma_0))^d$ into $(H^{-1}(\Omega))^d \times (L^2(\Omega))^d$ by

$$L\mathbf{v} = \left\{-\frac{\partial \mathbf{y}}{\partial t}(T;\mathbf{v}), \mathbf{y}(T;\mathbf{v})\right\}. \tag{7.18}$$

We define next F_1 and F_2 by

$$F_1(\mathbf{v}) = \frac{1}{2}\int_{\Sigma_0} |\mathbf{v}|^2 \,\mathrm{d}\Sigma, \tag*{$(7.19)_1$}$$

$$F_2(\mathbf{f}^0, \mathbf{f}^1) = \left\{\begin{array}{l}0 \text{ if } \mathbf{f}^0 \in -\mathbf{z}^1 + \beta_1 B_{-1}, \quad \mathbf{f}^1 \in \mathbf{z}^0 + \beta_0 B, \\ +\infty \text{ otherwise.}\end{array}\right. \tag*{$(7.19)_2$}$$

Problem (7.17) is then equivalent to

$$\inf_{\mathbf{v} \in (L^2(\Sigma_0))^d} [F_1(\mathbf{v}) + F_2(L\mathbf{v})]. \tag{7.20}$$

By duality, we obtain

$$\inf_{\mathbf{v}\in(L^2(\Sigma_0))^d}[F_1(\mathbf{v})+F_2(L\mathbf{v})] = -\inf_{\hat{\mathbf{f}}\in(H_0^1(\Omega))^d\times(L^2(\Omega))^d}[F_1^*(L^*\hat{\mathbf{f}})+F_2^*(-\hat{\mathbf{f}})]. \tag{7.21}$$

The operator L^* is defined as follows. We introduce $\hat{\varphi}$, $\hat{\psi}$ solution of

$$\begin{cases} \dfrac{\partial^2\hat{\varphi}}{\partial t^2} - \Delta\hat{\varphi} + \alpha\boldsymbol{\nabla}\dfrac{\partial\hat{\psi}}{\partial t} = 0 \text{ in } Q, \\ -\dfrac{\partial\hat{\psi}}{\partial t} - \Delta\hat{\psi} - \alpha\boldsymbol{\nabla}\cdot\hat{\varphi} = 0 \text{ in } Q, \end{cases} \tag{7.22$_1$}$$

$$\hat{\varphi}(T) = \hat{\mathbf{f}}^0 \in (H_0^1(\Omega))^d, \quad \frac{\partial\hat{\varphi}}{\partial t}(T) = \hat{\mathbf{f}}^1 \in (L^2(\Omega))^d, \quad \hat{\psi}(T) = 0, \tag{7.22$_2$}$$

$$\hat{\varphi} = \mathbf{0}, \quad \hat{\phi} = 0 \text{ on } \Sigma. \tag{7.22$_3$}$$

Then if $\hat{\mathbf{f}} = \{\hat{\mathbf{f}}^0, \hat{\mathbf{f}}^1\}$, we have

$$L^*\hat{\mathbf{f}} = \frac{\partial\hat{\varphi}}{\partial n} \text{ on } \Sigma_0. \tag{7.23}$$

We obtain thus as dual problem (i.e. for the minimization problem in the right-hand side of (7.21))

$$\inf_{\hat{\mathbf{f}}}\left[\frac{1}{2}\int_{\Sigma_0}\left|\frac{\partial\hat{\varphi}}{\partial n}\right|^2 \mathrm{d}\Sigma + \langle\mathbf{z}^1,\hat{\mathbf{f}}^0\rangle \right.$$
$$\left. -\int_\Omega \mathbf{z}^0\cdot\hat{\mathbf{f}}^1\,\mathrm{d}x + \beta_1\|\hat{\mathbf{f}}^0\|_{(H_0^1(\Omega))^d} + \beta_0\|\hat{\mathbf{f}}^1\|_{(L^2(\Omega))^d}\right] \tag{7.24}$$

Remark 7.9. The same considerations apply to the Neumann controls (i.e. of type (7.5)).

7.4. Approximate controllability via penalty

We consider again (7.1)–(7.4) (with $\theta_0 = 0$) and we introduce (with obvious notation):

$$J_k(\mathbf{v}) = \frac{1}{2}\int_{\Sigma_0}|\mathbf{v}|^2\,\mathrm{d}\Sigma + \frac{k_0}{2}\|\mathbf{y}(T;\mathbf{v})-\mathbf{z}^0\|_{L^2}^2 + \frac{k_1}{2}\|\frac{\partial\mathbf{y}}{\partial t}(T;\mathbf{v})-\mathbf{z}^1\|_{H^{-1}}^2. \tag{7.25}$$

In (7.25) we have

$$k = \{k_0, k_1\}, \quad k_i > 0, \quad k_i \text{ ‘large’ for } i = 0, 1. \tag{7.26}$$

The control problem

$$\inf_{\mathbf{v}\in(L^2(\Sigma_0))^d} J_k(\mathbf{v}) \tag{7.27}$$

has a *unique* solution, $\mathbf{u}_k$.

Considerations similar to those of Section 6, apply; thus we shall have

$$\mathbf{y}(T;\mathbf{u}_k) \in \mathbf{z}^0 + \beta_0 B, \quad \frac{\partial \mathbf{y}}{\partial t}(T;\mathbf{u}_k) \in \mathbf{z}^1 + \beta_1 B_{-1}, \tag{7.28}$$

for k 'large enough', the 'large enough' not being defined in a constructive way.

In order to obtain estimates on the choice of k, we will now consider the *dual problem* of (7.25). We consider again, therefore, the operator L: $(L^2(\Sigma_0))^d \to (H^{-1}(\Omega))^d \times (L^2(\Omega))^d$, defined by

$$L(v) = \left\{-\frac{\partial \mathbf{y}}{\partial t}(T;v), \mathbf{y}(T;v)\right\}$$

and we introduce

$$F_3(\mathbf{f}) = \tfrac{1}{2}k_0\|\mathbf{f}^1 - \mathbf{z}^0\|^2_{L^2} + \tfrac{1}{2}k_1\|\mathbf{f}^0 + \mathbf{z}^1\|^2_{H^{-1}}. \tag{7.29}$$

With $F_1(\cdot)$ still defined by $(7.19)_1$, we clearly have

$$\inf_{\mathbf{v}\in(L^2(\Sigma_0))^d} J_k(\mathbf{v}) = \inf_{\mathbf{v}\in(L^2(\Sigma_0))^d} [F_1(\mathbf{v}) + F_3(L\mathbf{v})]. \tag{7.30}$$

It follows by duality that

$$\inf_{\mathbf{v}\in(L^2(\Sigma_0))^d} J_k(\mathbf{v}) = -\inf_{\mathbf{f}}[F_1^*(L^*\mathbf{f}) + F_3^*(-\mathbf{f})] \tag{7.31}$$

with $\mathbf{f} = \{\mathbf{f}_0, \mathbf{f}_1\} \in (H_0^1(\Omega))^d \times (L^2(\Omega))^d$ in (7.31).

After some calculations, we obtain

$$\begin{aligned}\inf_{\mathbf{v}} J_k(\mathbf{v}) \quad = \quad & -\inf_{\mathbf{f}}\left[\frac{1}{2}\int_{\Sigma_0}\left|\frac{\partial\varphi}{\partial n}\right|^2 \mathrm{d}\Sigma + \frac{1}{2k_0}\|\mathbf{f}^1\|^2_{L^2} - \int_\Omega \mathbf{z}^0\cdot\mathbf{f}^1\,\mathrm{d}x\right.\\ & \left. + \frac{1}{2k_1}\|\mathbf{f}^0\|^2_{H_0^1} + \langle\mathbf{z}^0, \mathbf{f}^1\rangle\right].\end{aligned} \tag{7.32}$$

The dual problem to the control problem (7.27) *is therefore*

$$\inf_{\mathbf{f}}\left[\frac{1}{2}\int_{\Sigma_0}\left|\frac{\partial\varphi}{\partial n}\right|^2 \mathrm{d}\Sigma + \frac{1}{2k_0}\|\mathbf{f}^1\|^2_{L^2} - \int_\Omega \mathbf{z}^0\cdot\mathbf{f}^1\,\mathrm{d}x + \frac{1}{2k_1}\|\mathbf{f}^0\|^2_{H_0^1} + \langle\mathbf{z}^1, \mathbf{f}^0\rangle\right], \tag{7.33}$$

with $\mathbf{f} = \{\mathbf{f}^0, \mathbf{f}^1\} \in (H_0^1(\Omega))^d \times (L^2(\Omega))^d$ in (7.33).

Let us denote by $\mathbf{f}_k$ the solution of (7.33); it is characterized (with obvious notation) by

$$\begin{aligned}&\mathbf{f}_k = \{\mathbf{f}_k^0, \mathbf{f}_k^1\} \in (H_0^1(\Omega))^d \times (L^2(\Omega))^d,\\ &\int_{\Sigma_0}\frac{\partial\varphi_k}{\partial n}\cdot\frac{\partial\varphi}{\partial n}\,\mathrm{d}\Sigma + \frac{1}{k_0}\int_\Omega \mathbf{f}_k^1\cdot\mathbf{f}^1\,\mathrm{d}x + \frac{1}{k_1}\int_\Omega \nabla\mathbf{f}_k^0\cdot\nabla\mathbf{f}^0\,\mathrm{d}x\\ &\quad = \int_\Omega \mathbf{z}^0\cdot\mathbf{f}^1\,\mathrm{d}x - \langle\mathbf{z}^1, \mathbf{f}^0\rangle\\ &\forall\mathbf{f} = \{\mathbf{f}^0, \mathbf{f}^1\} \in (H_0^1(\Omega))^d \times (L^2(\Omega))^d.\end{aligned} \tag{7.34}$$

Similarly, the solution $\mathbf{f}_\beta$ of problem (7.24) is characterized by the following *variational inequality*

$$\mathbf{f}_\beta = \{\mathbf{f}_\beta^0, \mathbf{f}_\beta^1\} \in (H_0^1(\Omega))^d \times (L^2(\Omega))^d$$

$$\int_{\Sigma_0} \frac{\partial \varphi_\beta}{\partial n} \cdot \frac{\partial}{\partial n}(\varphi - \varphi_\beta)\,\mathrm{d}\Sigma + \beta_1(\|\mathbf{f}^0\|_{H_0^1} - \|\mathbf{f}_\beta^0\|_{H_0^1}) + \beta_0(\|\mathbf{f}^1\|_{L^2} - \|\mathbf{f}_\beta^1\|_{L^2})$$
$$\geq \int_\Omega \mathbf{z}^0 \cdot (\mathbf{f}^1 - \mathbf{f}_\beta^1)\,\mathrm{d}x - \langle \mathbf{z}^1, \mathbf{f}^0 - \mathbf{f}_\beta^0 \rangle,$$
$$\forall \mathbf{f} \in (H_0^1(\Omega))^d \times (L^2(\Omega))^d. \tag{7.35}$$

Taking $\mathbf{f} = \mathbf{f}_k$ in (7.34) (respectively $\mathbf{f} = \mathbf{0}$ and $\mathbf{f} = 2\mathbf{f}_\beta$ in (7.35)) we obtain

$$\int_{\Sigma_0} \left|\frac{\partial \varphi_k}{\partial n}\right|^2 \mathrm{d}\Sigma + \frac{1}{k_0}\int_\Omega |\mathbf{f}_k^1|^2\,\mathrm{d}x + \frac{1}{k_1}\int_\Omega |\boldsymbol{\nabla} \mathbf{f}_k^0|^2\,\mathrm{d}x$$
$$= \int_\Omega \mathbf{z}^0 \cdot \mathbf{f}_k^1\,\mathrm{d}x - \langle \mathbf{z}^1, \mathbf{f}_k^0 \rangle, \tag{7.36}$$

$$\int_{\Sigma_0} \left|\frac{\partial \varphi_\beta}{\partial n}\right|^2 \mathrm{d}\Sigma + \beta_0\|\mathbf{f}_\beta^1\|_{L^2} + \beta_1\|\mathbf{f}_\beta^0|_{H_0^1} = \int_\Omega \mathbf{z}^0 \cdot \mathbf{f}_\beta^1\,\mathrm{d}x - \langle \mathbf{z}^1, \mathbf{f}_\beta^0 \rangle. \tag{7.37}$$

Assuming that problems (7.34), (7.35) have the same solution, it follows from (7.36), (7.37) that

$$\frac{1}{k_0}\|\mathbf{f}_k^1\|_{L^2}^2 + \frac{1}{k_1}\|\mathbf{f}_k^1\|_{H_0^1}^2 = \beta_0\|\mathbf{f}_k^1\|_{L^2} + \beta_1\|\mathbf{f}_k^0\|_{H_0^1}, \tag{7.38}$$

which suggests the following simple (may be too simple) rule: adjust k_0, k_1 so that

$$\|\mathbf{f}_k^1\|_{L^2}^2 k_0^{-1} = \beta_0, \quad \|\mathbf{f}_k^0\|_{H_0^1}^2 k_1^{-1} = \beta_1. \tag{7.39}$$

We plan *numerical experiments* to validate (7.39).

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